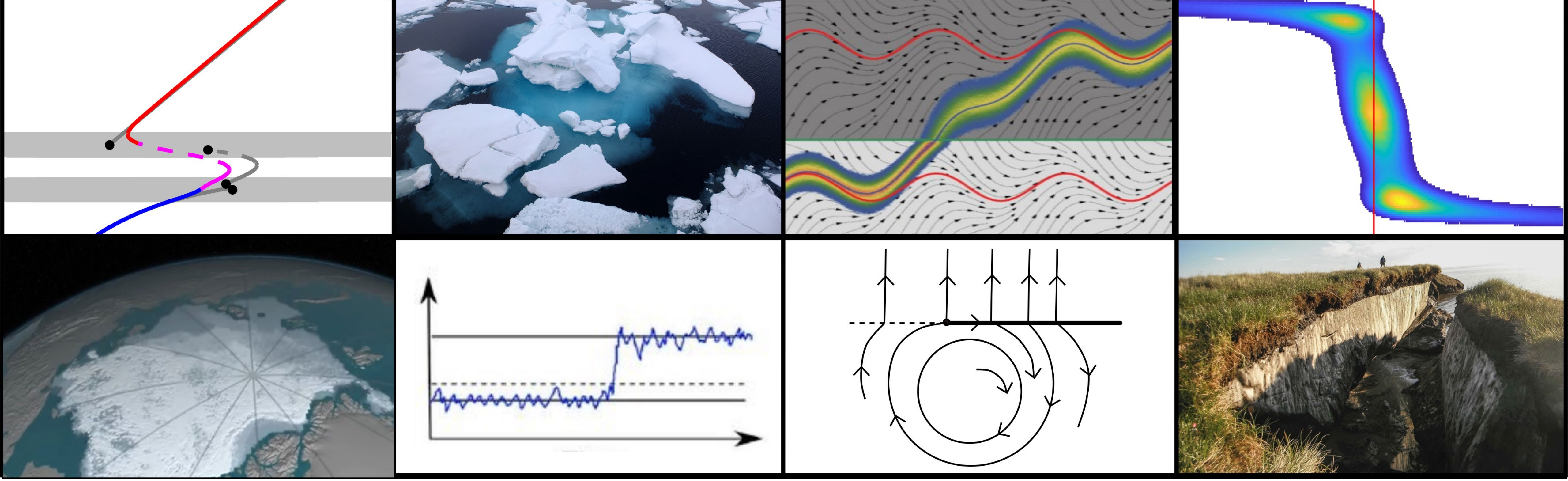


Most probable transition paths in piecewise-smooth stochastic differential equations

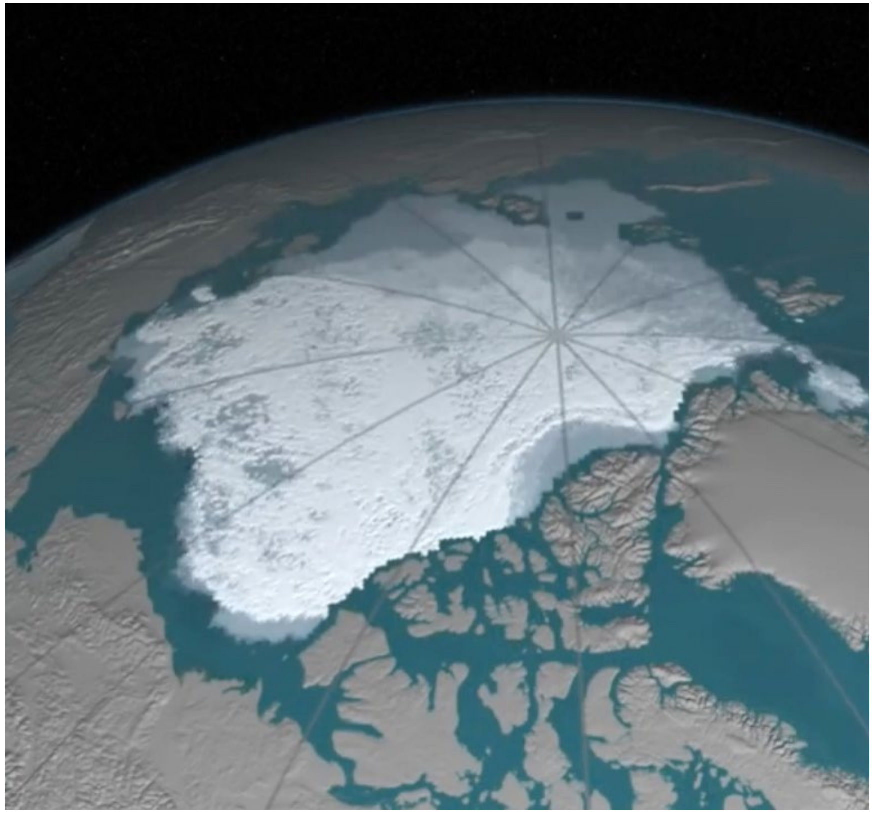
Kaitlin Hill
Wake Forest University
(Fall 2022: St. Mary's University)

April 6, 2022

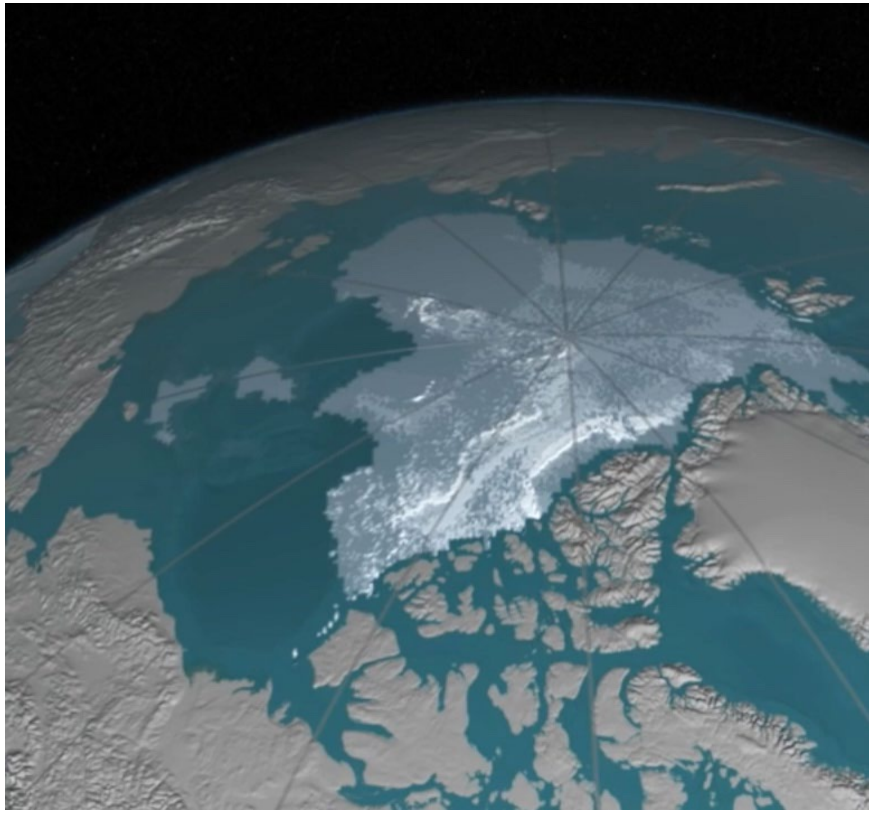
Joint Mathematics Meetings

Tipping in piecewise-smooth systems: Arctic sea ice

Sept 1986



Sept 2016



(NASA Scientific Visualization Studio)

Tipping in piecewise-smooth systems: Arctic sea ice

$$\frac{dE}{dt} = \boxed{\big(1 - \alpha(E)\big)F_s(t)} - \boxed{(F_l(t) + BT(E,t))}$$

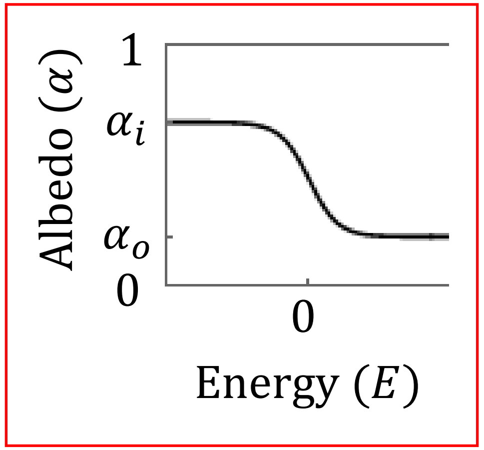
incoming energy   outgoing energy

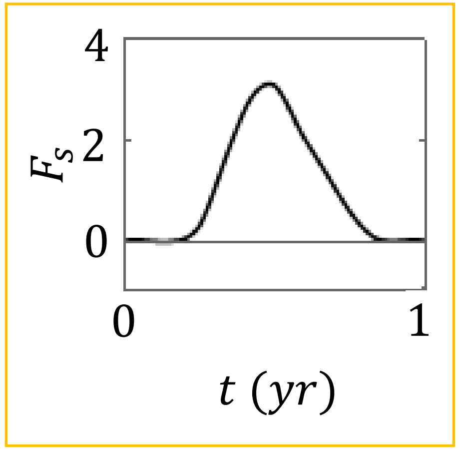
(Eisenman and Wettlaufer 2009)

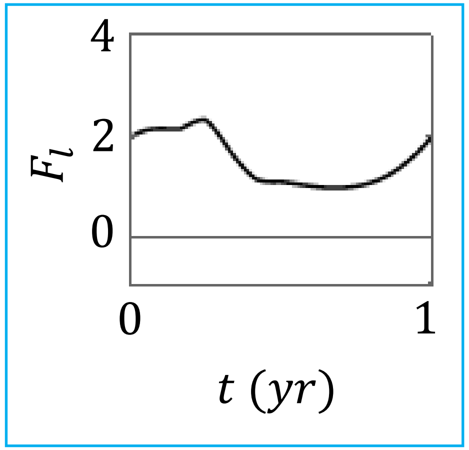
Arctic energy balance model

$$\frac{dE}{dt} = \big(1 - \alpha(E)\big)F_s(t) - \big(F_l(t) + BT(E,t)\big)$$

albedo — incoming solar energy — outgoing energy — temperature



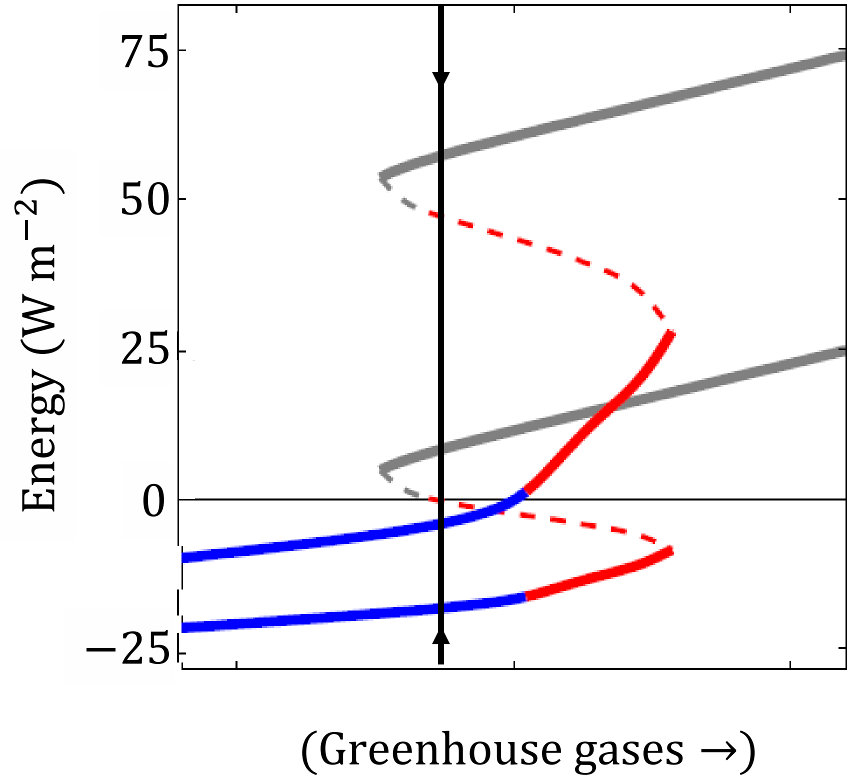




$$T \propto \begin{cases} E & \text{(ocean)} \\ 0 & \text{(melt)} \\ f(t,E) & \text{(ice)} \end{cases}$$

(Eisenman and Wettlaufer 2009)

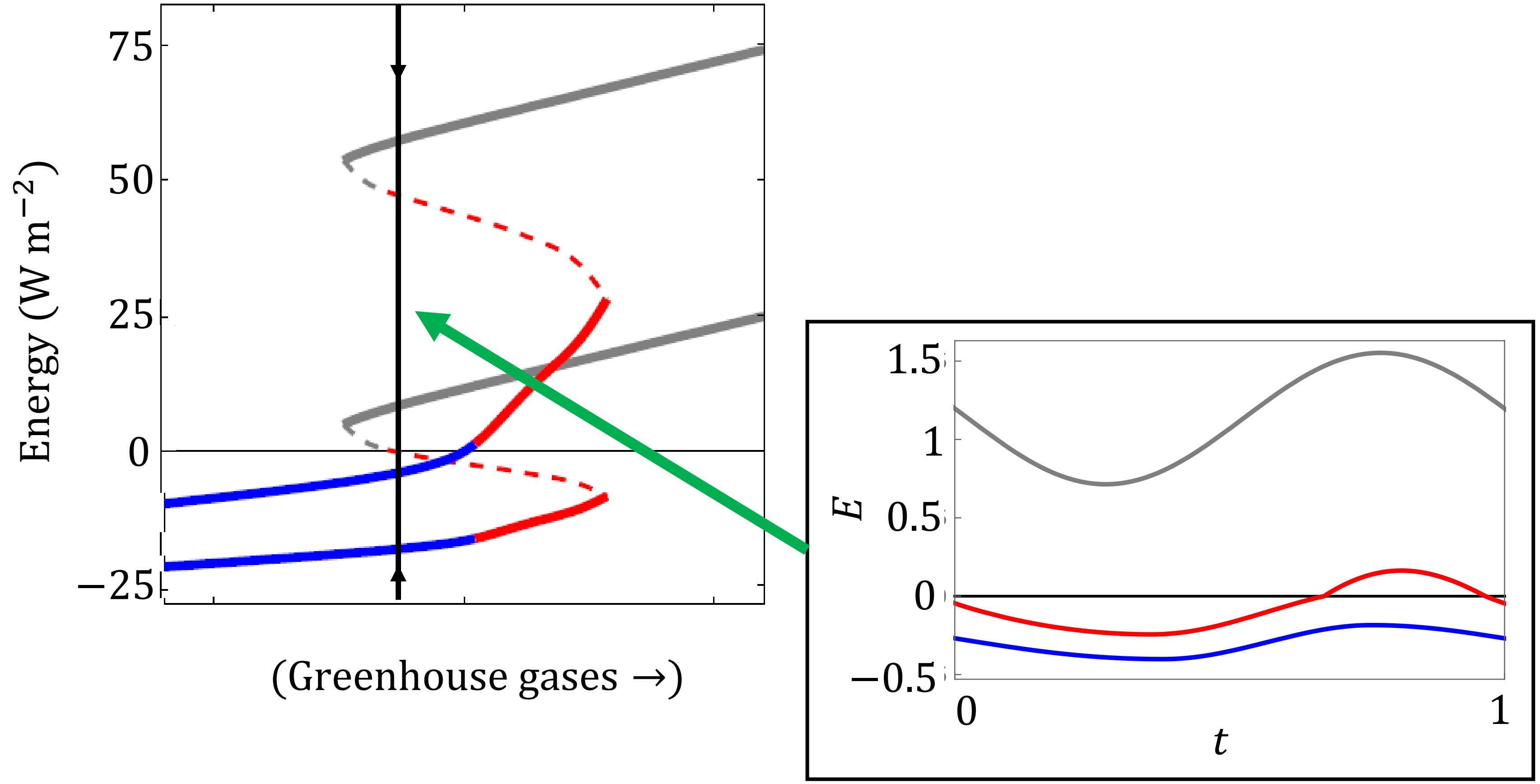
Motivation: what will the transition be like?



- Perennially ice-free
- Seasonally ice-free
- Perennially ice-covered

(Eisenman and Wettlaufer 2009)

Motivation: what will the transition be like?



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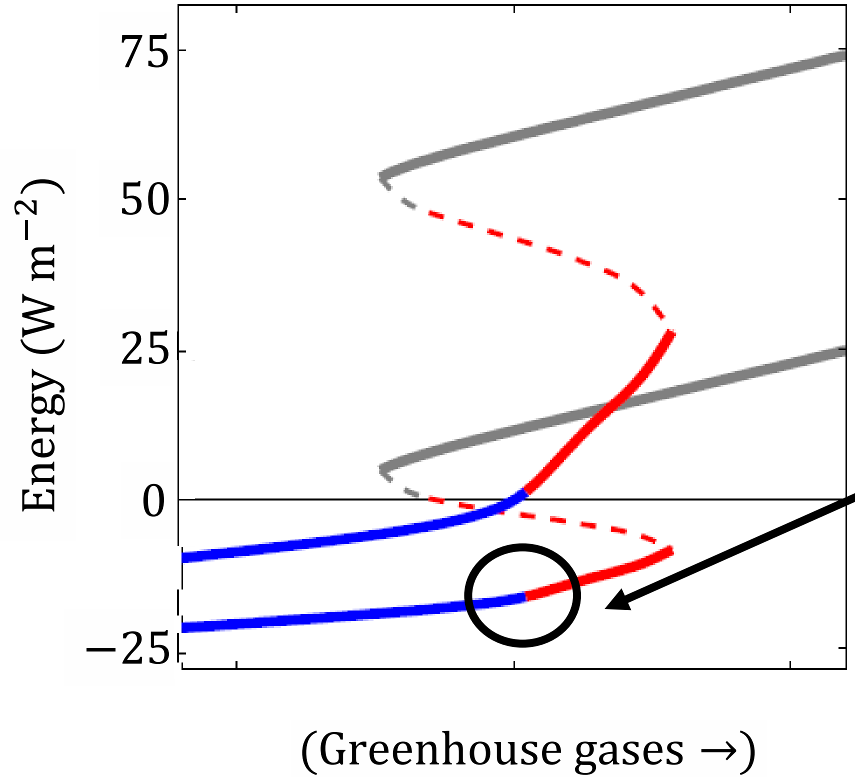


No bifurcation-induced 'tip' from current state to seasonally ice-free state

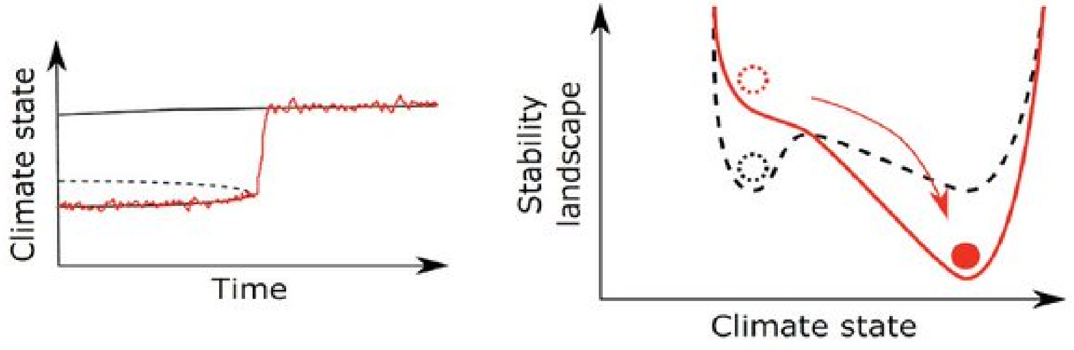
- Perennially ice-free
- Seasonally ice-free
- Perennially ice-covered

(Eisenman and Wettlaufer 2009)

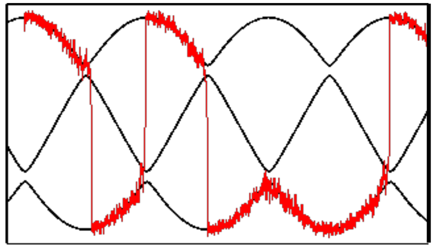
Mathematical descriptions of tipping points

Many possible ways to mathematically describe tipping:

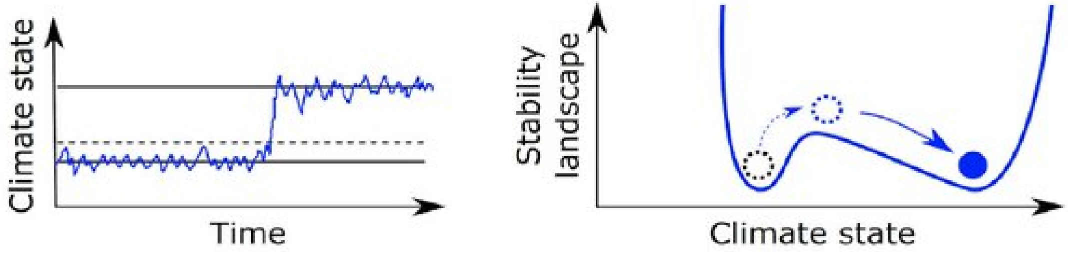
Bifurcation-induced tipping



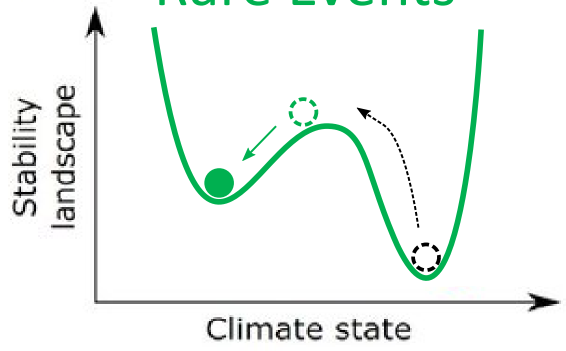
Stochastic resonance



Noise-induced tipping

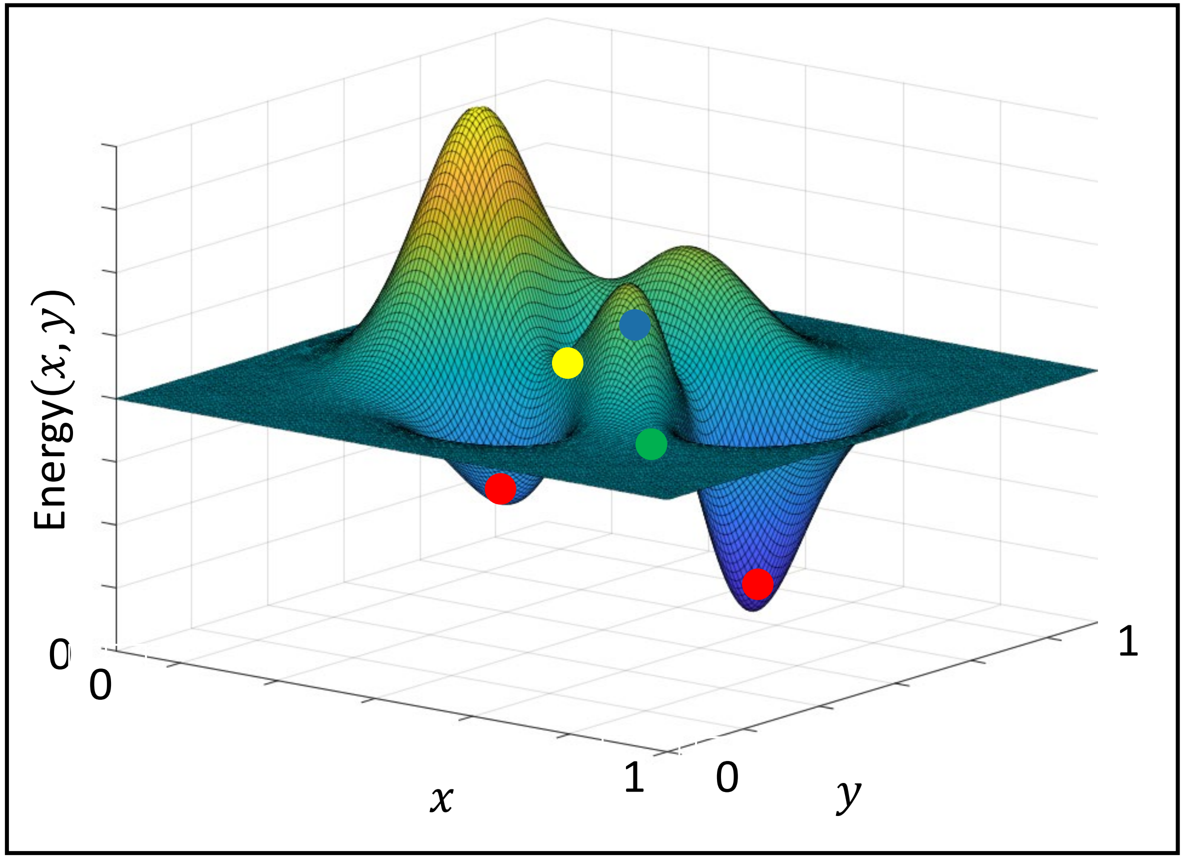


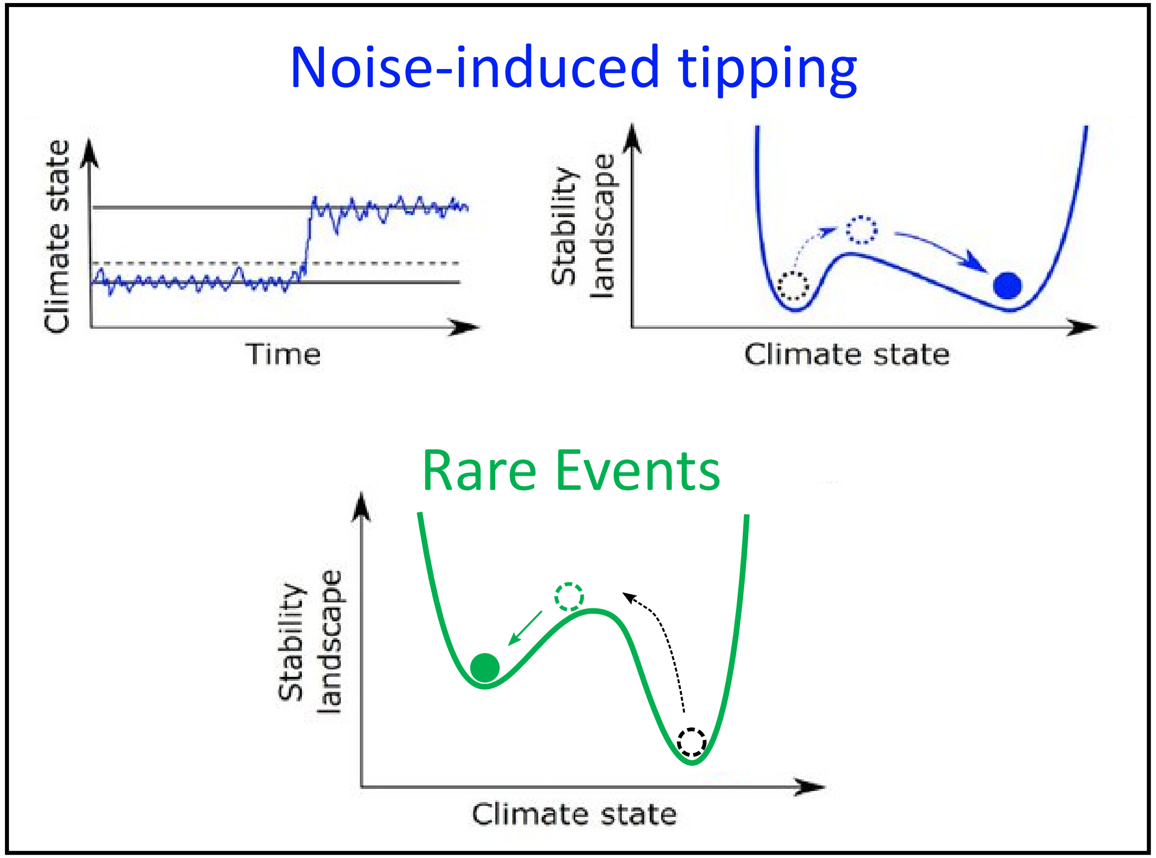
Rare Events



(IPCC AR6)

Noise-induced tipping: rare events



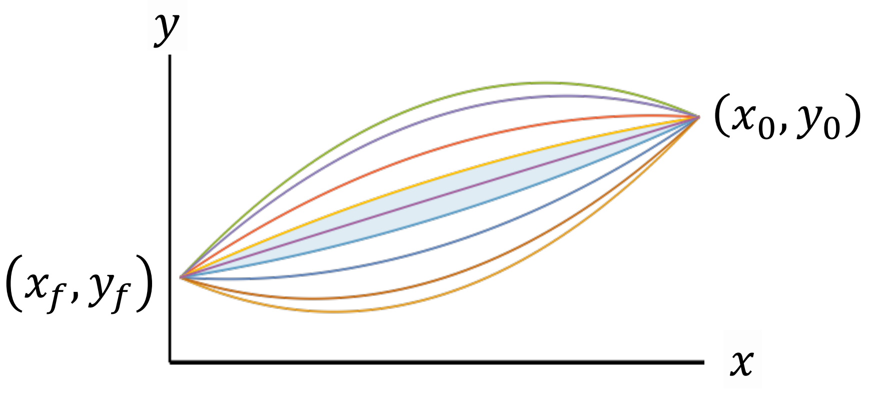


Freidlin-Wentzell theory of large deviations: Smooth systems

Consider $\boldsymbol{F}: \mathbb{R}^n \to \mathbb{R}^n$, so that

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) \quad \xrightarrow{\textit{noise}} \quad d\boldsymbol{x}_t = \boldsymbol{F}(\boldsymbol{x})\mathrm{dt} + \sigma dW_t$$

where the noise is normally distributed.



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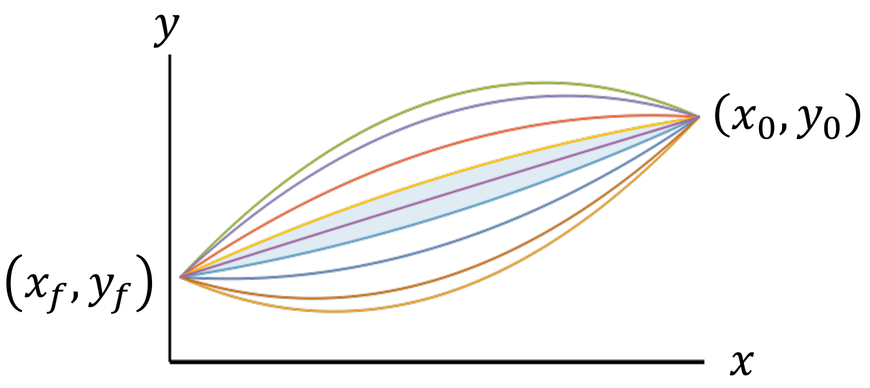

The probability density function describing all possible paths $\boldsymbol{\alpha}(t)$ is

$$P(\boldsymbol{\alpha}) \propto \exp\left[-\frac{1}{\sigma^2}\underbrace{\int_{t_0}^{t_f} \|\dot{\boldsymbol{\alpha}} - \boldsymbol{F}\|^2\, dt}_{\text{Freidlin-Wentzell}} + \sigma^2 \underbrace{\int_{t_0}^{t_f} \nabla \cdot \boldsymbol{F}(x, y)\, dt}_{\text{Onsager-Machlup}}\right]$$

Freidlin-Wentzell theory of large deviations: Smooth systems

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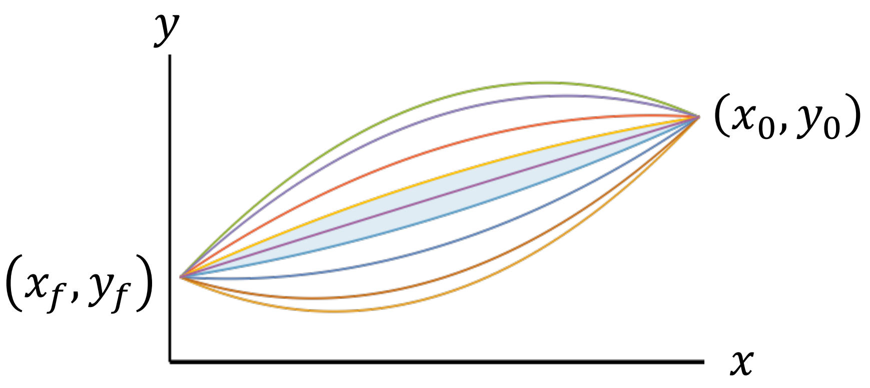

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The most probable path is the maximum of $P(\boldsymbol{\alpha})$, or the minimum of the rate functional

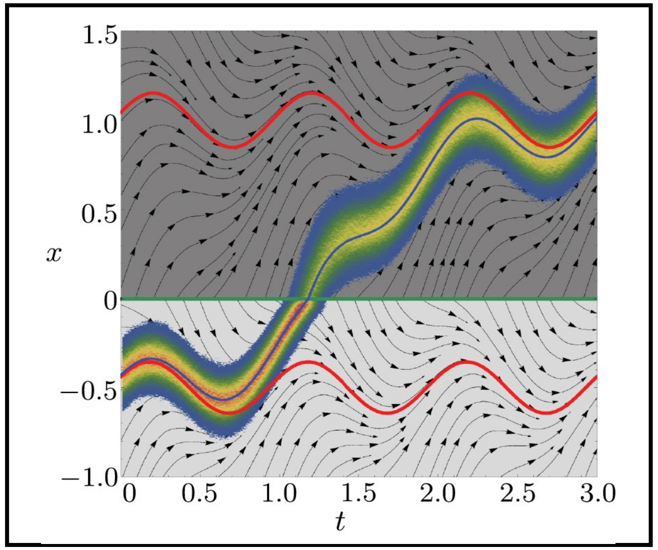
$$I[\boldsymbol{\alpha}(t)] = \int_{t_0}^{t_f} \|\dot{\boldsymbol{\alpha}} - \boldsymbol{F}\|^2\, dt$$

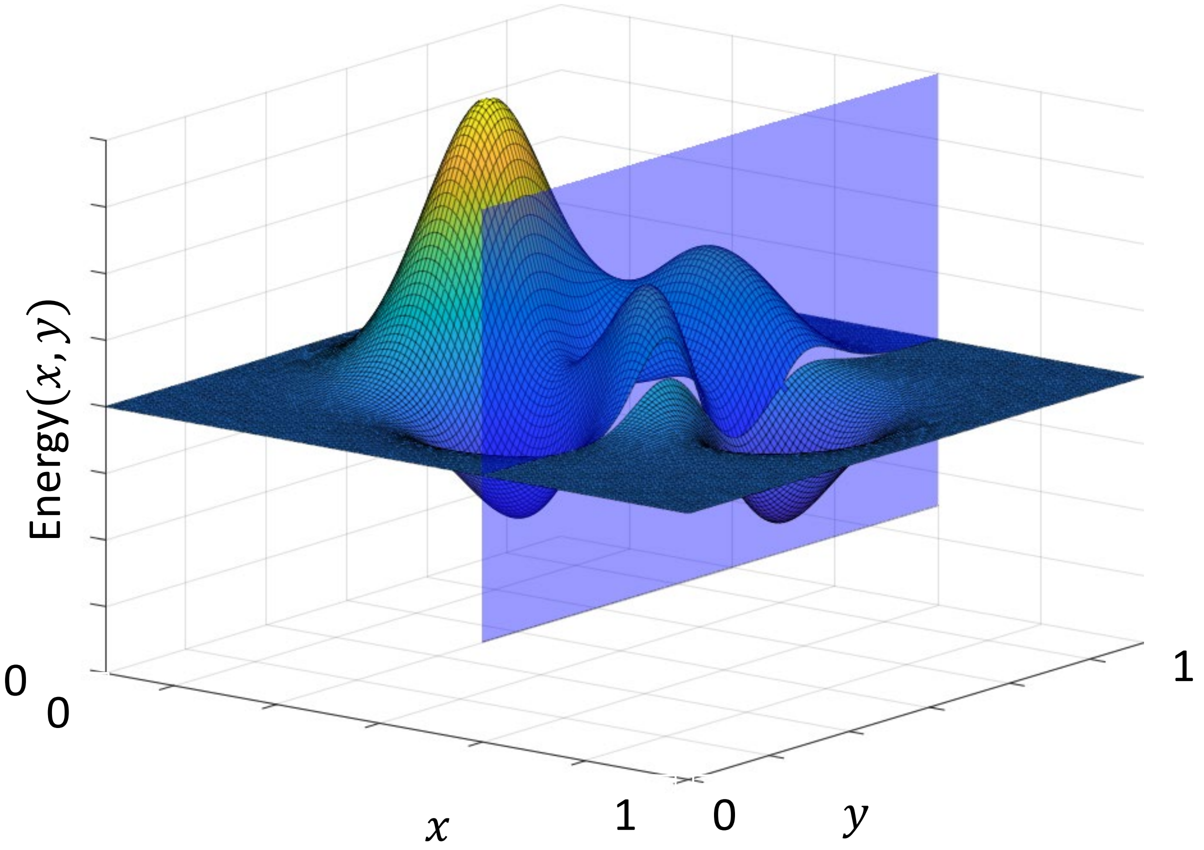
In piecewise-smooth systems

Problem:

We can't describe rare event tipping for piecewise-smooth systems!

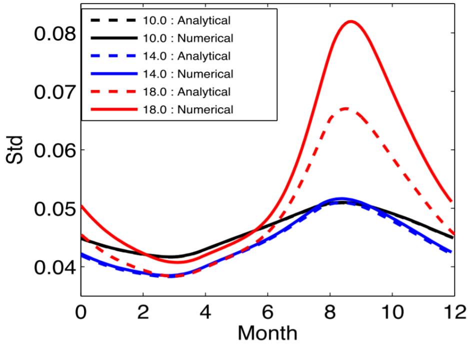
We need $\boldsymbol{F}$ *to be smooth (differentiable)*.





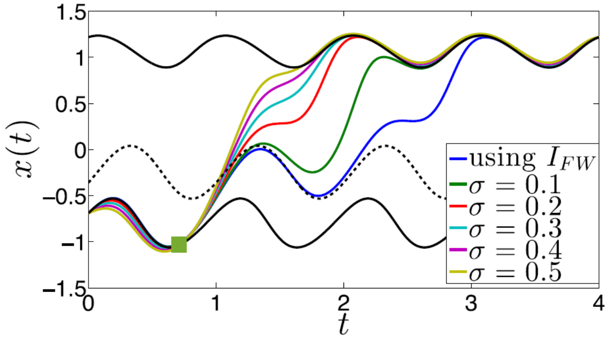
Investigations of tipping in periodically-forced SDEs

Sea ice model, numerical approach



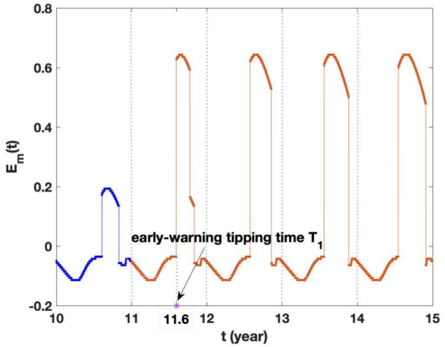
(Moon and Wettlaufer, 2017)

Periodically-forced, smooth, general SDE



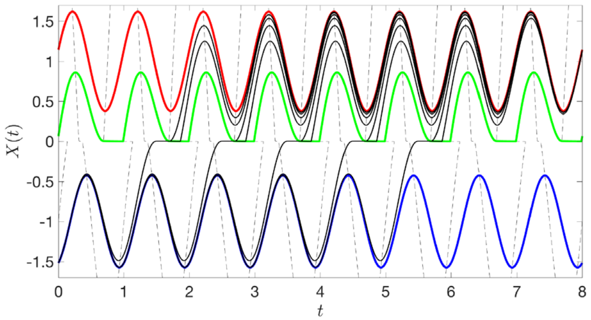
(Chen, Silber, Gemmer, Volkening, 2019)

Sea ice model, α −Levy noise



(Fang, …, Wiggins, 2020)

Piecewise-smooth general SDE



(Zanetell, 2018)

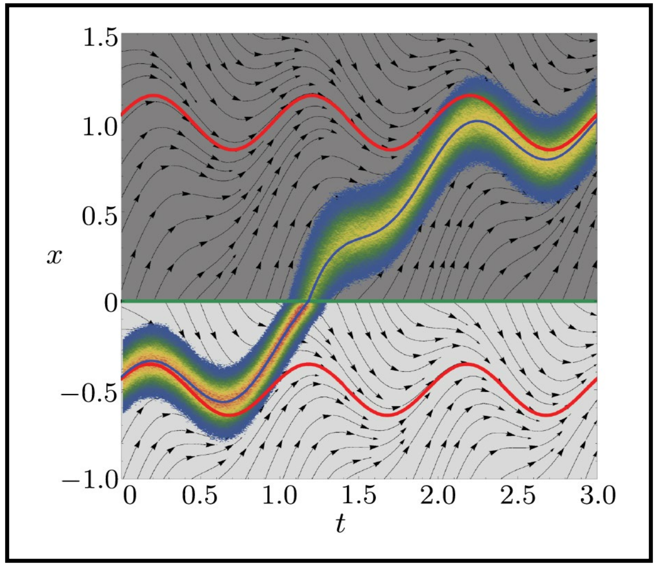
Theorem: Most probable paths in piecewise-smooth systems

Consider a piecewise-smooth system,

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(x, \boldsymbol{y}), \qquad \boldsymbol{F}(x, \boldsymbol{y}) = \begin{cases} \boldsymbol{F}^+(x, \boldsymbol{y}), & x > 0 \\ \boldsymbol{F}^-(x, \boldsymbol{y}), & x < 0 \end{cases}$$

where $\boldsymbol{x} = (x, \boldsymbol{y}) \in \mathbb{R}^n$ and $\boldsymbol{F} = (F_1, \boldsymbol{G}) : \mathbb{R}^n \to \mathbb{R}^n$

The most probable path for a piecewise-smooth system is $\boldsymbol{\alpha} = (\alpha, \boldsymbol{\beta})$, the minimum of the rate functional



$$\bar{I}[\boldsymbol{\alpha}(t)] = \int_{t \in \{t : x \neq 0\}} \|\dot{\boldsymbol{\alpha}} - \boldsymbol{F}\|^2 \, dt$$

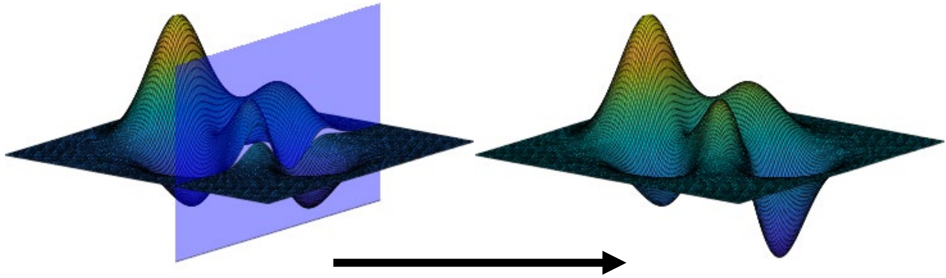
$$+ \int_{t \in \{t : x = 0\}} \min_{\lambda \in [0,1]} \left\{ (\lambda F_1^+ + (1-\lambda) F_1^-)^2 + \left(\dot{\boldsymbol{\beta}} - \lambda \boldsymbol{G}^+ - (1-\lambda)\boldsymbol{G}^- \right)^2 \right\} dt$$

(KH, Zanetell, Gemmer, *submitted*)

Ideas for proof

Consider

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(x, \boldsymbol{y}), \qquad \boldsymbol{F}(x, \boldsymbol{y}) = \begin{cases} \boldsymbol{F}^{+}(x, \boldsymbol{y}), & x > 0 \\ \boldsymbol{F}^{-}(x, \boldsymbol{y}), & x < 0 \end{cases}$$



where $\boldsymbol{x} = (x, \boldsymbol{y}) \in \mathbb{R}^n$

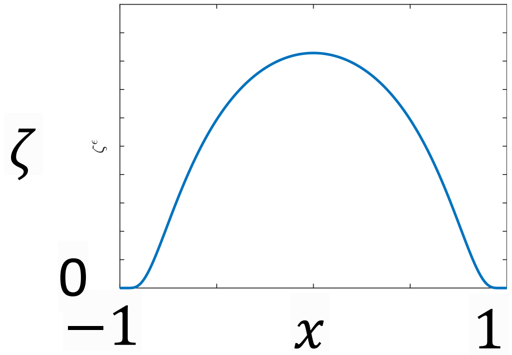
Smooth $\boldsymbol{F}$ using a convolution with $\zeta^{\epsilon}(x) = \zeta(x/\epsilon)/\epsilon$:

$$\boldsymbol{F}^{\epsilon} = \zeta^{\epsilon}(x) * \boldsymbol{F}(x, \boldsymbol{y}) = \int_{-\infty}^{0} \zeta^{\epsilon}(x - s)\boldsymbol{F}^{-}(s, \boldsymbol{y}) \; ds + \int_{0}^{\infty} \zeta^{\epsilon}(x - s)\boldsymbol{F}^{+}(s, \boldsymbol{y}) \; ds$$

Most probable paths for $\boldsymbol{F}^{\epsilon}$ are minimizers of

$$I_{\epsilon}[\boldsymbol{\alpha}] = \int_{t_0}^{t_f} \|\dot{\boldsymbol{\alpha}} - \boldsymbol{F}^{\epsilon}\|^2 \, dt$$

Freidlin-Wentzell



Γ-Convergence ensures convergence in PWS limit

We need to derive the appropriate functional to minimize in the piecewise-smooth limit.

Definition: (Γ-Convergence)
A functional $I_\epsilon[\boldsymbol{\alpha}]$ $\Gamma -$ converges to another functional $\bar{I}[\boldsymbol{\alpha}]$ with respect to H^1 weak convergence if, for all $\boldsymbol{\alpha}$,

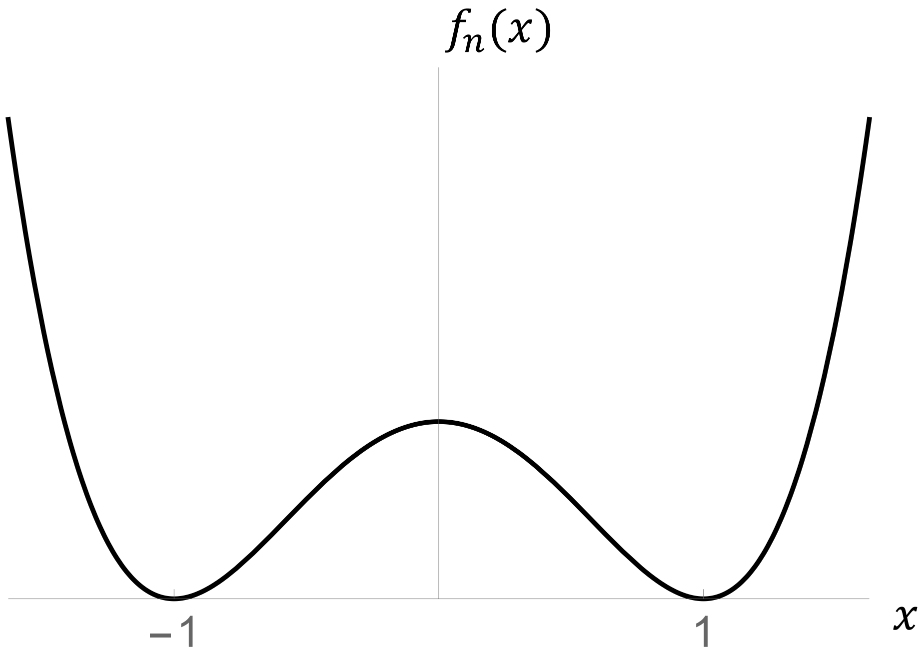
1. (*recovery sequence*)
   There exists a sequence $\boldsymbol{\alpha}_\epsilon$ satisfying $\boldsymbol{\alpha}_\epsilon \to \boldsymbol{\alpha}$ weakly in H^1 such that
   $$\lim_{\epsilon \to 0} I_\epsilon[\boldsymbol{\alpha}_\epsilon] = \bar{I}[\boldsymbol{\alpha}]$$
2. (*liminf inequality*)
   For every sequence $\boldsymbol{\alpha}_\epsilon$ satisfying $\boldsymbol{\alpha}_\epsilon \to \boldsymbol{\alpha}$ weakly in H^1,
   $$\bar{I}[\boldsymbol{\alpha}] \leq \liminf I_\epsilon[\boldsymbol{\alpha}_\epsilon]$$

Takeaways:

- Guarantees minimizers of I_ϵ converge to minimizers of $\bar{I}$
- $\bar{I}$ is convex and has a minimum

$\Gamma -$ limit example

For functions on $\mathbb{R}^n$, the $\Gamma -$ limit is the lower semicontinuous envelope.



$\Gamma - \lim_{n \to \infty} f_n$

$\Gamma -$ limit example

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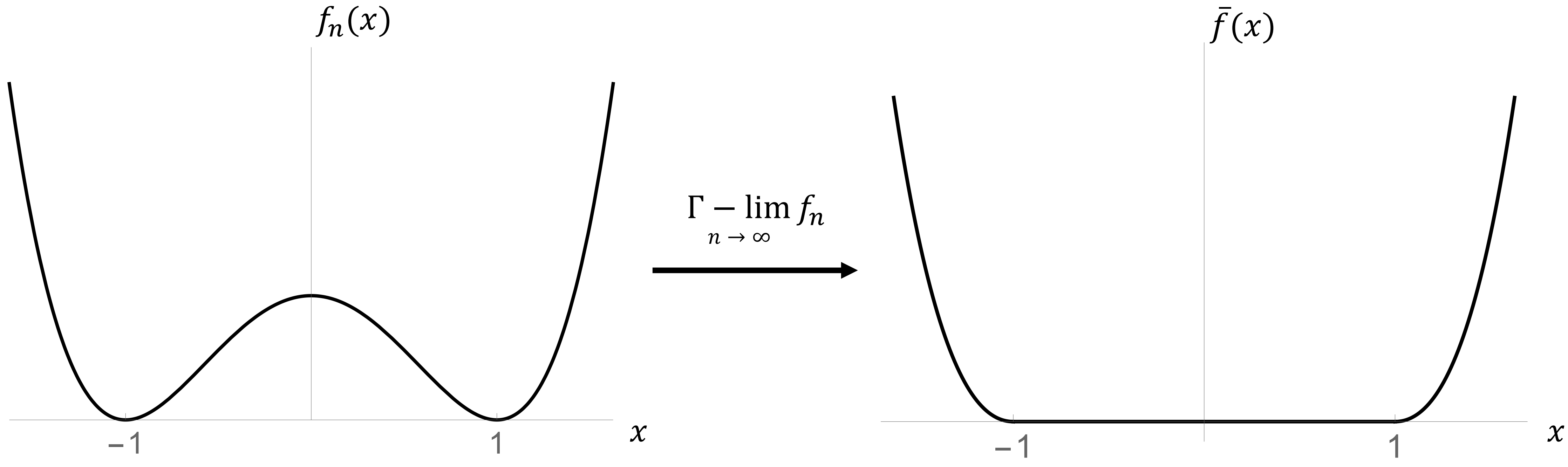

Theorem: Practical takeaways

Consider a piecewise-smooth system,

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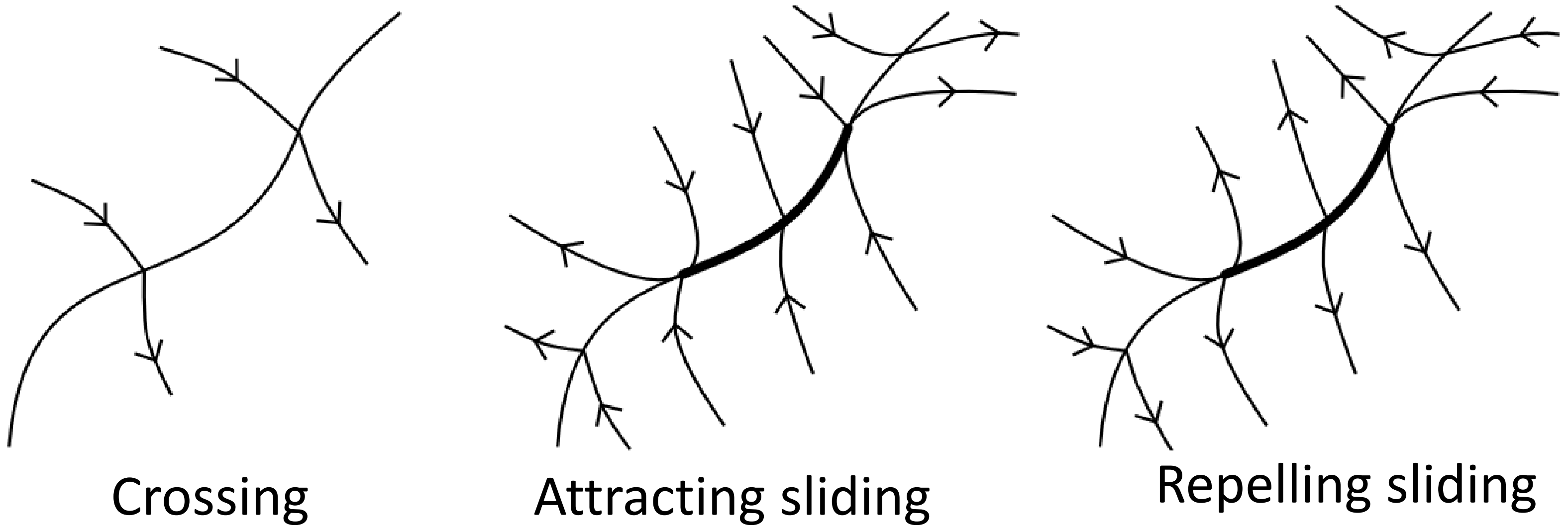

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The most probable path for a piecewise-smooth system is $\boldsymbol{\alpha} = (\alpha,\boldsymbol{\beta})$, the minimum of the rate functional

$$\bar{I}[\boldsymbol{\alpha}(t)] = \int_{t\,\in\,\{t\,:\,x\neq 0\}} \|\dot{\boldsymbol{\alpha}} - \boldsymbol{F}\|^2\,dt$$

$$+ \int_{t\,\in\,\{t\,:\,x=0\}} \min_{\lambda\in[0,1]} \left\{(\lambda F_1^{+} + (1-\lambda)F_1^{-})^2 + \left(\dot{\boldsymbol{\beta}} - \lambda \boldsymbol{G}^{+} - (1-\lambda)\boldsymbol{G}^{-}\right)^2\right\} dt$$

Zero when:

- Sliding in a sliding region

Nonzero when:

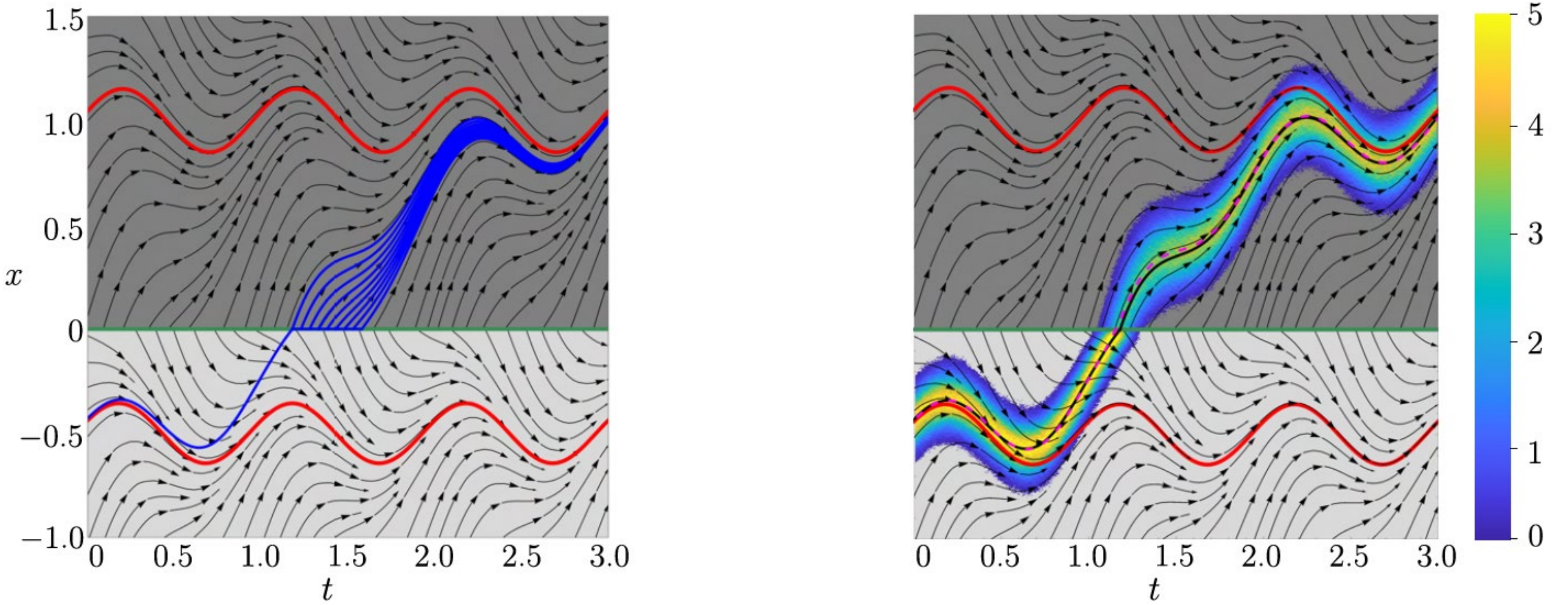
- Sliding in a crossing region
- Sliding oppositely to the convex combination:

$$\dot{\beta} = \lambda G^{+} - (1-\lambda)G^{-}$$

Case study: Periodically-forced system

Consider the system:

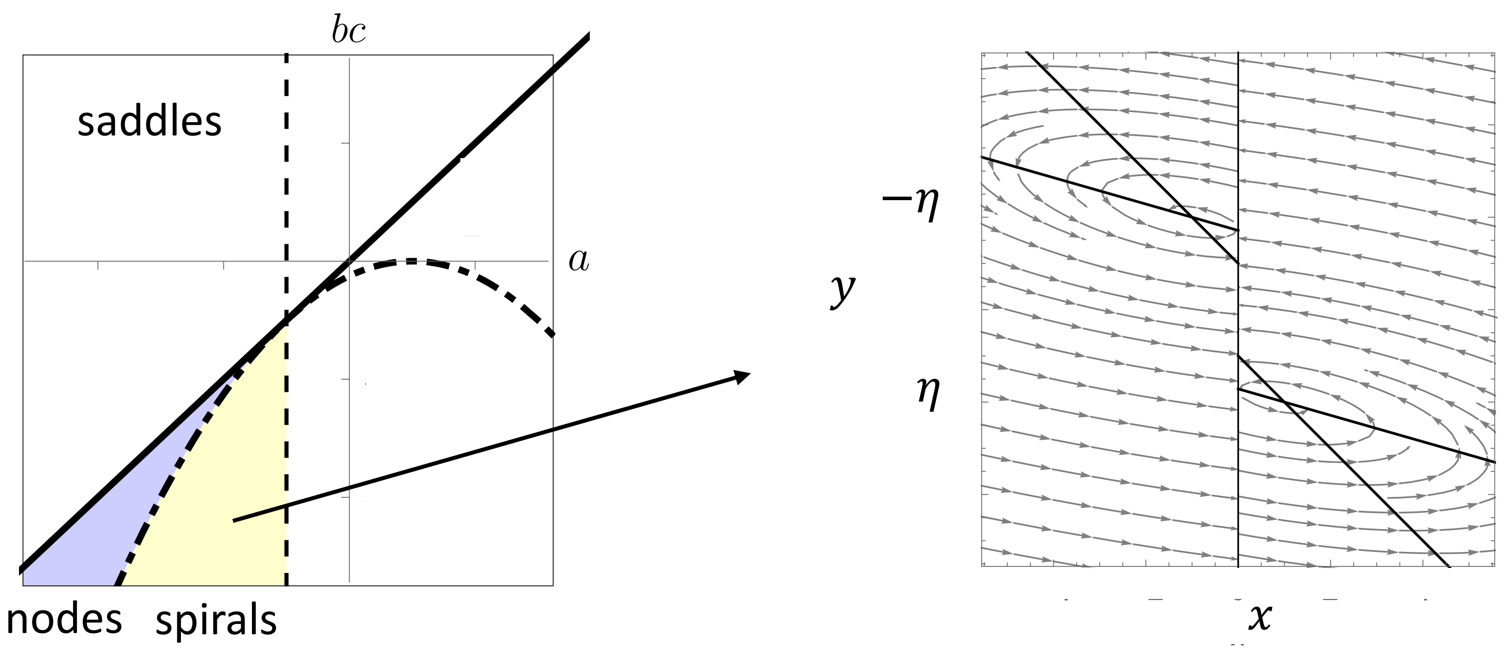
$$\dot{x} = \begin{cases} -r_+(x-1) + A_+ \cos(2\pi t)\,, & x > 0 \\ -r_-(x-a) + A_- \cos(2\pi(t-p))\,, & x < 0 \end{cases}$$



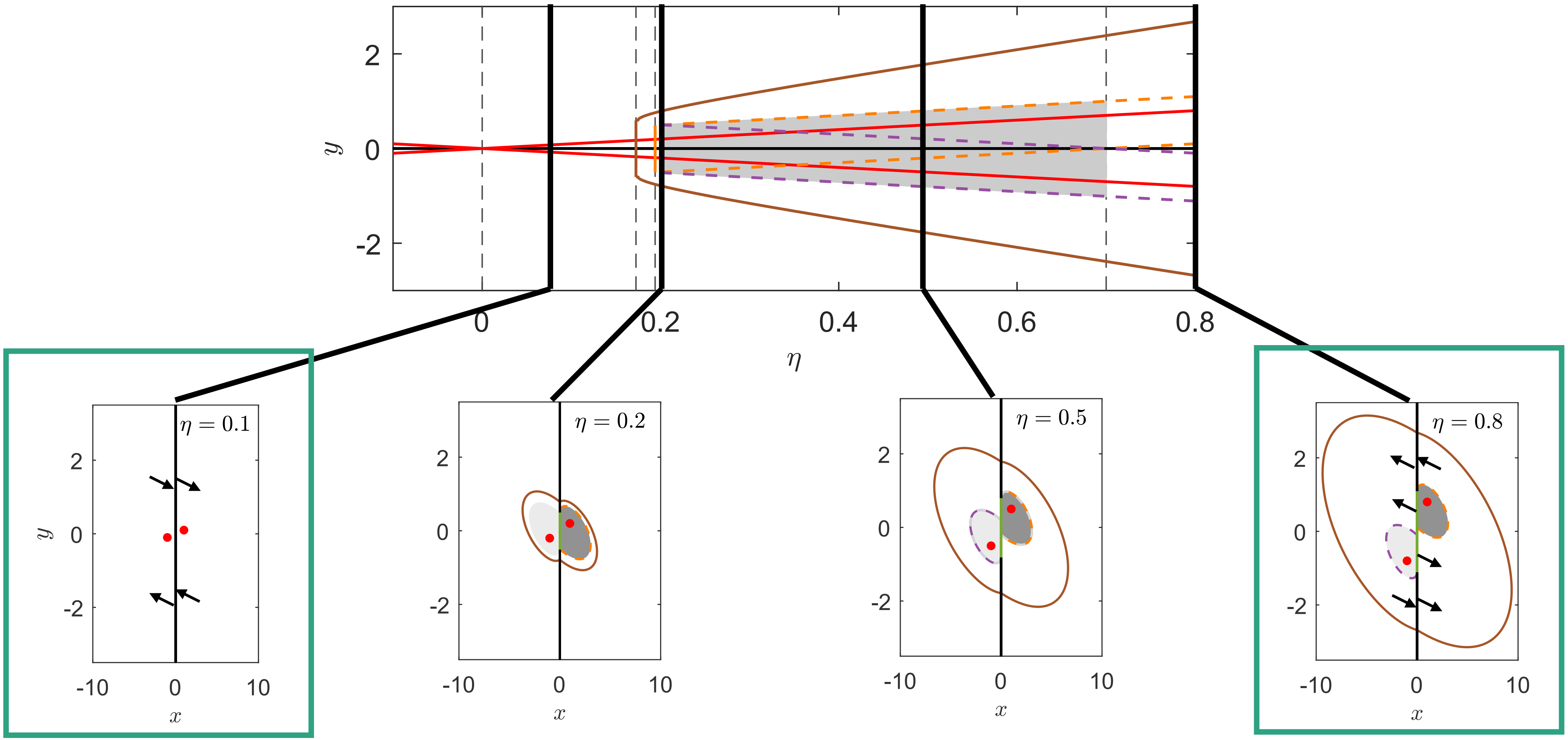
Case study: Planar system

Consider the system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a(x-1)+b(y-\eta) \\ c(x-1)+y-\eta \end{pmatrix}, & x>0 \\ \begin{pmatrix} p(x+1)+q(y+\eta) \\ r(x+1)+y+\eta \end{pmatrix}, & x<0 \end{cases}$$



Case study: Solution behavior



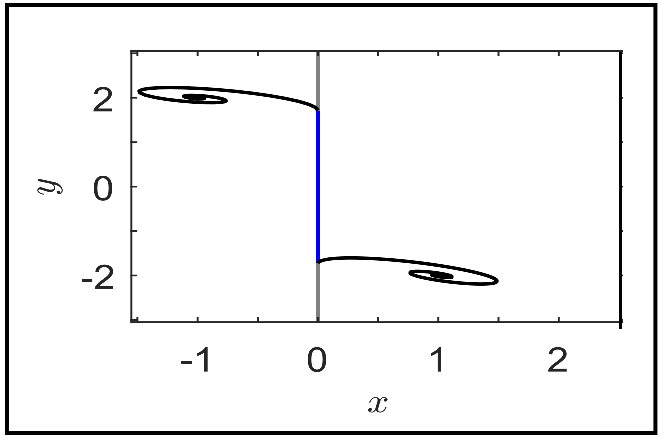
Case study: most probable path(s)

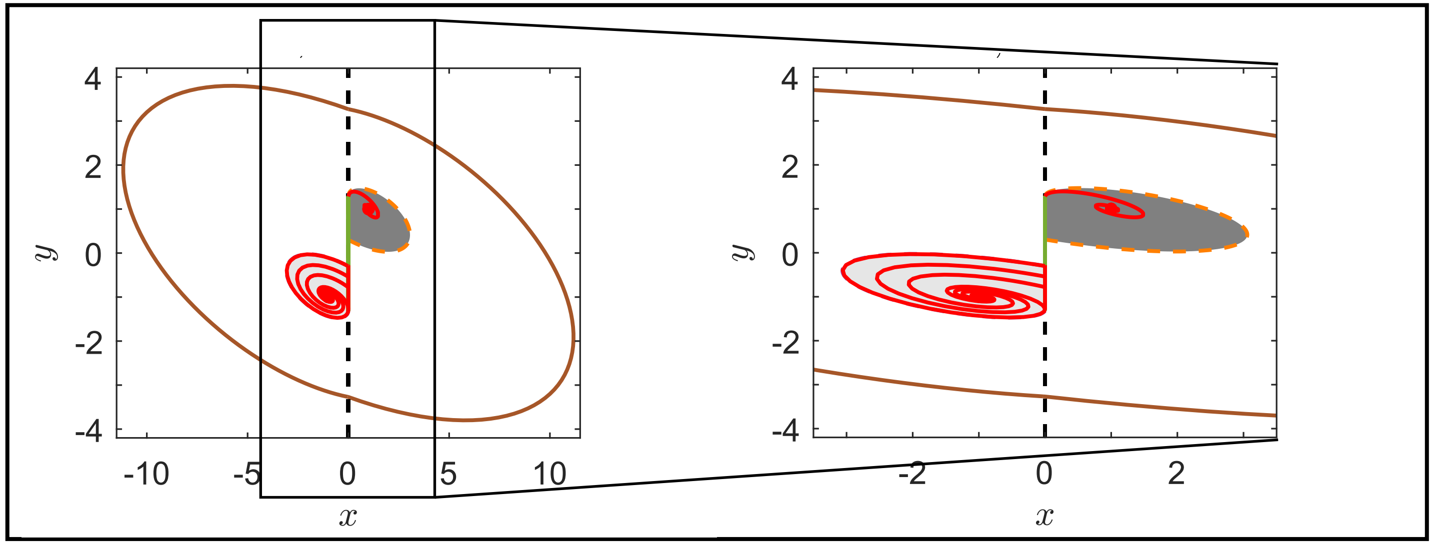
Minimizing

$$\bar{I}[\alpha(t),\beta(t)] = \int_{t\,\in\{t\,:\,x\neq 0\}} \left(\dot{\alpha} - f(\alpha,\beta)\right)^2 + \left(\dot{\beta} - g(\alpha,\beta)\right)^2 dt$$

$$+ \int_{t\,\in\{t\,:\,x=0\}} \min_{\lambda\in[0,1]} \left\{(\lambda f^+ + (1-\lambda) f^-)^2 + \left(\dot{\beta} - \lambda g^+ - (1-\lambda) g^-\right)^2\right\} dt$$

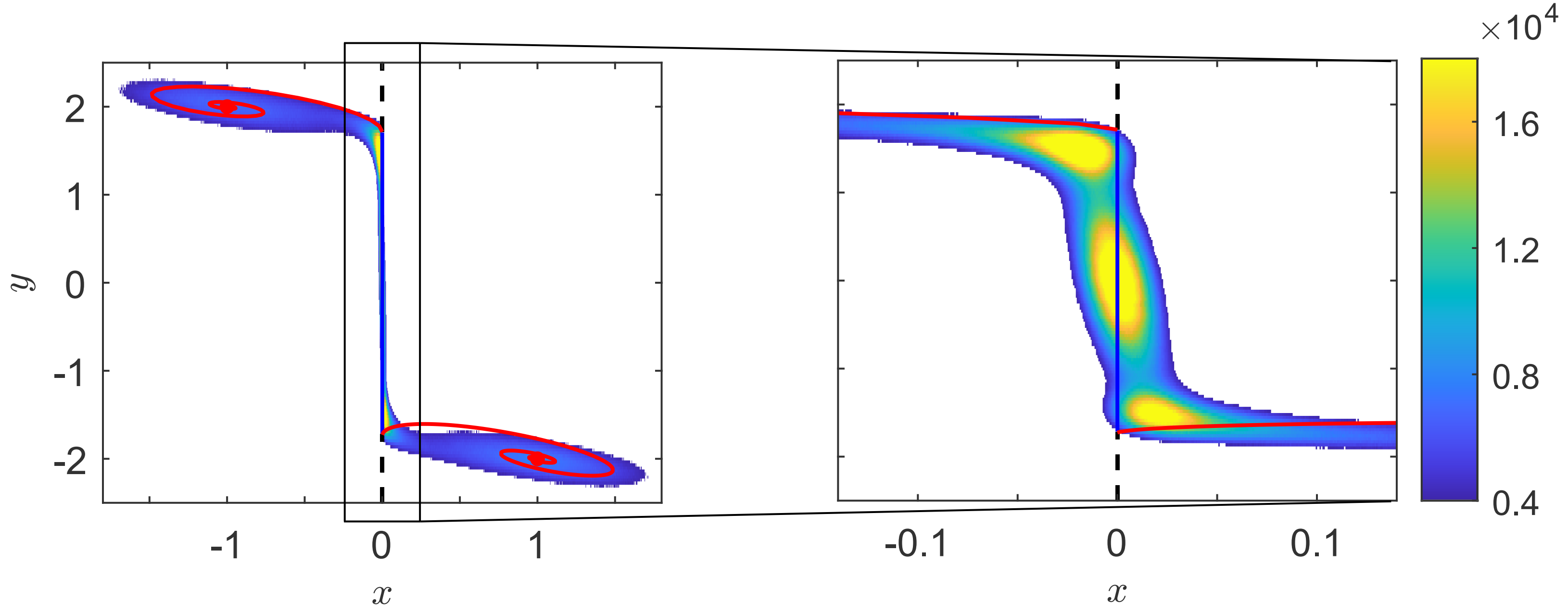
predicts





Validation: Monte Carlo simulations

$$\dot{x} = f(x,y) \qquad dx_t = f(x,y)\mathrm{dt} + \sigma_x dW_t$$
$$\dot{y} = g(x,y) \qquad dy_t = g(x,y)\mathrm{dt} + \sigma_y dW_t$$



$N = 1.056 \times 10^7$ simulations
$n = 6.487 \times 10^4$ tips

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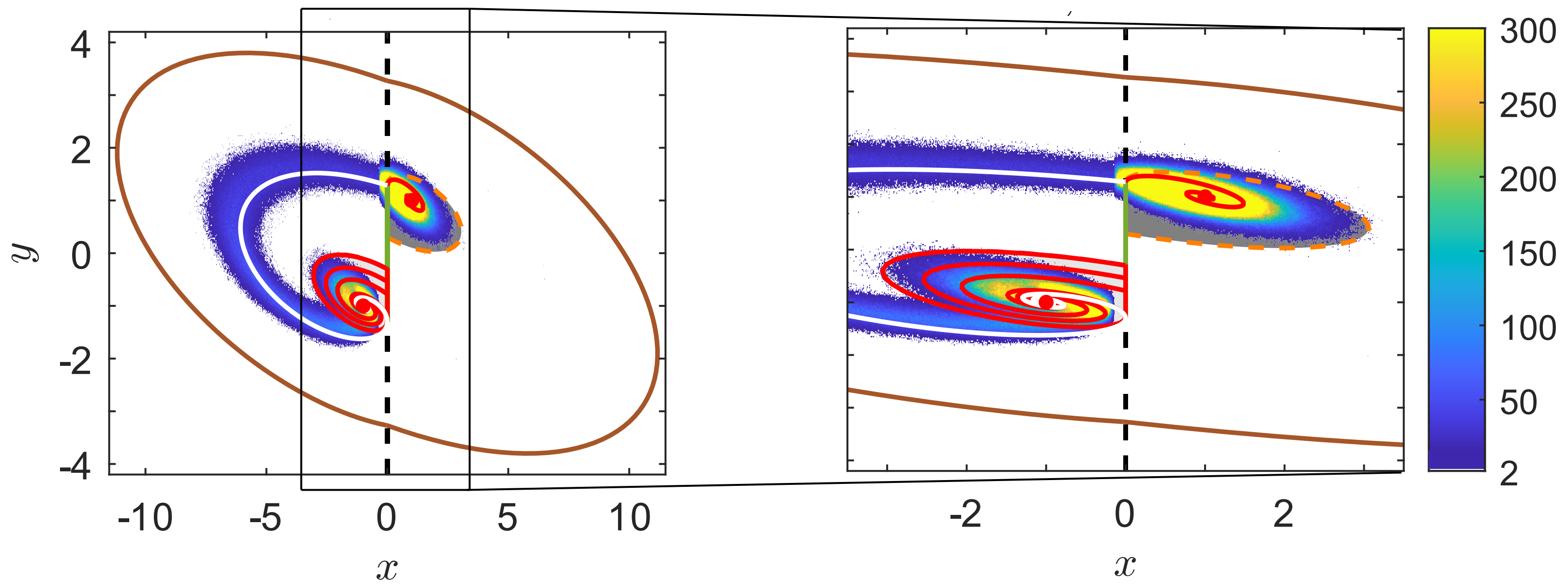


$N = 1.676 \times 10^7$ simulations
$n = 1.960 \times 10^4$ tips

Most probable path does not match the predicted path!

Current and future work

Applications! Some models that may include switches:

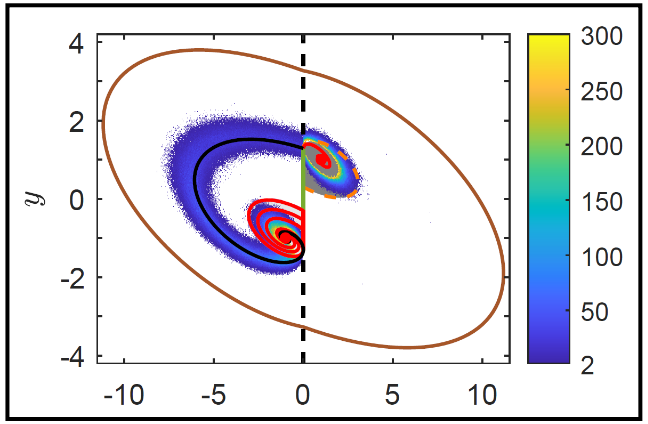
Arctic sea ice

RC circuits

Neuron firing

Predator-prey / Population ecology

Epidemics



Onsager-Machlup term in the rate functional

Contributions to the limiting functional for repelling sliding

Thanks for coming!

Many thanks to collaborators:

John Gemmer, Wake Forest U

Jessica Zanetell, Wake Forest U