## UMAP

## Modules in Undergraduate Mathematics and Its <br> Applications

## Published in cooperation with

The Society for Industrial and Applied Mathematics,

The Mathematical Association of America,

The National Council of Teachers of Mathematics,

The American Mathematical Association of Two-Year Colleges,

The Institute for Operations Research and the Management Sciences, and

The American
Statistical Association.

## Module 817

Math for Poets

Rachel Wells Hall



Combinatorics
TITLE: Math for Poets

AUTHOR:

MATHEMATICAL FIELD:

APPLICATION FIELDS:

TARGET AUDIENCE:

ABSTRACT:

Rachel Wells Hall

Dept. of Mathematics St. Joseph's University 5600 City Avenue Philadelphia, PA 19131
http://www.sju.edu/~rhall
rhall@sju.edu

Combinatorics.

Poetry, music, culture of India

Students in a liberal arts course in mathematics.

This Module presents musical and mathematical topics in a general education mathematics course entitled "Sounding Number: Music and Mathematics from Ancient to Modern Times." Music has many connections to mathematics. The Greeks discovered that chords with a pleasing sound are related to simple ratios of integers. It is less well known that the rhythms of music and poetry have been studied mathematically. In ancient India, scholars first discovered Pascal's triangle and the Fibonacci numbers in the rhythms of poetry. This Module explores some mathematics related to rhythm in poetry and music, with extensions to mathematical topics such as binary codes and de Bruijn sequences. Exercises, worksheets, quizzes, and sample exam questions are included.

None.

The UMAP Journal 41 (2): 145-176. ©Copyright 2020 by COMAP, Inc. All rights reserved.

Permission to make digital or hard copies of part or all of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice. Abstracting with credit is permitted, but copyrights for components of this work owned by others than COMAP must be honored. To copy otherwise, to republish, to post on servers, or to redistribute to lists requires prior permission from COMAP.

COMAP, Inc., Suite 3B, 175 Middlesex Tpke., Bedford, MA 01730

## Math for Poets

Rachel Wells Hall
Dept. of Mathematics
St. Joseph's University 5600 City Avenue
Philadelphia, PA 19131
http://www.sju.edu/~rhall rhall@sju.edu
Table of Contents

1. Introduction ..... 1
2. Meter as Binary Code ..... 1
2.1 Problem 1: Listing Meters ..... 2
2.2 Problem 2: Counting Meters ..... 3
2.3 The Binary Number System ..... 4
3. Hemachandra-Fibonacci Numbers ..... 4
3.1 Recursion ..... 6
3.2 The Padovan Sequence ..... 6
4. The Expanding Mountain of Jewels ..... 7
4.1 Recursion and the Meruprastāra ..... 8
5. Naming Meters and ..... 9
de Bruijn Sequences ..... 9
5.1 Remembering Meters ..... 9
5.2 de Bruijn Sequences ..... 10
6. Patterns in Music and Architecture ..... 11
7. Solutions to the Exercises ..... 14
8. Worksheets ..... 17
8.1 Worksheet A: Rhythm Patterns of Fixed Duration ..... 17
8.2 Worksheet B: Musical Rhythms with 2- and 3-Beat Notes ..... 18
8.3 Worksheet C: The Expanding Mountain of Jewels ..... 19
8.4 Worksheet D: Patterns in the Meruprastāra ..... 20
9. Quizzes ..... 21
9.1 Quiz 1 ..... 21
9.2 Quiz 2 ..... 21
10. Solutions to Quizzes ..... 22
10.1 Solutions to Quiz 1 ..... 22
10.2 Solutions to Quiz 2 ..... 22
11. Sample Exam Questions ..... 23
12. Solutions to Sample Exam ..... 26
References ..... 28
Acknowledgments ..... 28
About the Author ..... 28

# Modules and Monographs in Undergraduate Mathematics and its Applications (UMAP) Project 

Paul J. Campbell<br>Solomon Garfunkel<br>Editor<br>Executive Director, COMAP

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

## 1. Introduction

But most by Numbers judge a Poet's Song,
And smooth or rough, with them, is right or wrong;
These Equal Syllables alone require, Tho' oft the Ear the open Vowels tire, While Expletives their feeble Aid do join, And ten low Words oft creep in one dull Line,
—Alexander Pope, An Essay on Criticism (1709)

Numerical patterns have fascinated humans for millennia: numbers that are powers of other numbers, squares that are sums of squares, numbers that form intriguing lists. This is the story of one of the earliest studies of rhythm, an investigation that led ancient Indian scholars to discover the mathematical patterns that Westerners know as the Fibonacci numbers, Pascal's triangle, and the binary counting system. Although our story initially concerns rhythm in poetry, the ancient Indians' ability and fascination with exploring rhythmic patterns also had a profound influence on their music.

## 2. Meter as Binary Code

In English, a poetic rhythm, called a meter, is a pattern of stressed and unstressed syllables. English poets use about a dozen different meters. Much poetry, including Shakespeare's plays, is written in iambic pentame-ter-five pairs of alternating unstressed and stressed syllables to a line. Alexander Pope's 700+-line iambic pentameter poem An Essay on Criticism (1709) is a good example. In the excerpt that begins this chapter, he ridicules critics who judge poetry "by numbers"-that is, solely on how well a poet follows strict metrical rules.

While English poets use relatively few meters, there are hundreds of different meters in Sanskrit, the classical language of India. Syllables in Sanskrit poetry are classified by duration (short or long) rather than stress. Any Sanskrit meter can be written as a binary code-a pattern of any length formed by two symbols. For example, there are eight binary codes of length 3 that are formed from the letters $L$ and $S$ :

LLL SLL LSL SSL LLS SLS LSS and SSS
These correspond to the eight three-syllable meters, using $S$ for a short syllable and $L$ for a long syllable.

Pingala, who probably lived in the last few centuries B.C., is thought to be the first Indian scholar to study meter mathematically. As is typical in ancient Indian literature, Pingala's writings took the form of short, cryptic verses, or sūtras, which served as memory aids for a larger set of concepts passed on orally. We are dependent on medieval commentators for transmission and interpretation of Pingala's writings.

Here are two of the questions that Pingala answered:

- What is a reliable way to list all the meters with a given number of syllables?
- How many meters have a given number of syllables?


### 2.1 Problem 1: Listing Meters.

There are a number of ways to solve this problem. Pingala's solution would result in the one-syllable meters listed as
L S
the two-syllable meters being listed as

$$
\text { LL } \quad \text { SL } \quad \text { LS } \quad \text { SS }
$$

and the three-syllable meters like this:

$$
\text { LLL } \quad \text { SLL } \quad \text { LSL } \quad \text { SSL } \quad \text { LLS } \quad \text { SLS } \quad \text { LSS } \quad \text { SSS }
$$

Here is how the four-syllable patterns would be listed:
LLLL, SLLL, LSLL, SSLL, LLSL, SLSL, LSSL, SSSL, LLLS, SLLS, LSLS, SSLS, LLSS, SLSS, LSSS, SSSS.
Several patterns are observable in these lists. The first column alternates L and S, the second alternates pairs of L's and pairs of S's, the third alternates four copies of the letters, and so on. There is symmetry in the lists, in the sense that the first pattern is equivalent to the last with the letters exchanged, and this is true for each pair of patterns that are at the same distance from the beginning and end.

In addition, there is a relationship between successive lists: for example, the list of four-syllable meters is formed from the list of three-syllable meters by first adding L's to the end of the list, then adding S's. This last observation is useful for describing an algorithm that will produce all meters of length $n$ in the order that Pingala did.

## Exercise

1. Write the list of five-syllable meters in the order that you think Pingala would have listed them. Describe the procedure that you followed to get the answer.

Theorem 1 (Listing $n$-syllable meters). The list of one-syllable patterns is $\{L, S\}$. Suppose that a list of $n$-syllable patterns is formed from a nonrepeating list of all $(n-1)$-syllable patterns by adding L's to the end of each $(n-1)$-syllable pattern, followed by the list resulting from adding $S$ 's to the end of each $(n-1)$-syllable pattern. Then each $n$-syllable pattern will occur exactly once in the new list.

Proof: It is clear that each of the one-syllable meters $(n=1)$ is listed once. Suppose that the algorithm results in each of the $(n-1)$-syllable patterns being listed exactly once. Use the algorithm to form a list of $n$-syllable patterns. Choose any $n$-syllable pattern. We want to show that any chosen pattern occurs exactly once in the new list.

- If the pattern ends in an $L$, then the algorithm shows that it appears exactly once in the first half of the list, its first $(n-1)$ syllables appear exactly once in the list of $(n-1)$-syllable patterns.
- If it ends in an $S$, it appears exactly once in the second half of the list for the same reason.

The proof essentially says that if the one-syllable patterns are correct, then the two-syllable patterns are correct, then the three-syllable patterns are correct, and so on, until your chosen length is reached-sort of like a row of dominoes falling down. This type of reasoning is called proof by induction. Although it may seem intuitively reasonable to argue this way, in fact, one of the axioms for the positive integers, the Axiom of Induction, legitimates such reasoning.

### 2.2 Problem 2: Counting Meters

How many meters have $n$ syllables? Counting the patterns on the lists above, you see the numbers $2,4,8$, and 16 , which equal $2^{1}, 2^{2}, 2^{3}$, and $2^{4}$. You might conjecture that there are 32 (or $2^{5}$ ) four-syllable meters, and, in general, there are $2^{n} n$-syllable meters. This is correct. It follows so closely from the theorem that mathematicians would call it a corollary, which is a theorem that may be proven from another theorem without much effort.
Corollary 1 (Counting $n$-syllable meters). The number of $n$-syllable meters is $2^{n}$.

## Exercises

2. How long is the list on which LLSSLSL appears?
3. Assuming that the theorem has been proven true, explain why the corollary is true. You don't need to write a formal proof, but use complete sentences.

### 2.3 The Binary Number System

Since there's nothing special about the letters L and S , the previous theorem and its corollary generalize to any set of binary codes. For example, there are $2^{5}$ patterns of length 5 that are formed from the letters $a$ and $b$. In some ways, Pingala anticipated the development of the binary number system. The binary number system is a base-two positional number system (our number system is a base-ten positional system). It has two digits, 0 and 1 , and its place values are powers of two-therefore, every number is also a binary code. The decimal numbers $1,2,8$, and 11 have binary representation $1,10,1000$, and 1011 , respectively. The binary number system was not fully described until Gottfried Leibniz did so in the 17th century.

## Exercises

4. Suppose that you flip a coin three times and write down the pattern of heads and tails, using H and T . The order of flips makes a differencethat is, HHT is different from HTH. How many patterns of three coin flips are there? List them and use your list to compute the likelihood of getting tails exactly once if you flip three times. Describe how you would list and count the patterns for any number of coin flips.
5. Here are the binary numbers from 8 (1000 in binary) to 15 (1111 in binary):

$$
\text { 1000, 1001, 1010, 1011, 1100, 1101, 1110, } 1111 .
$$

How is this sequence related to the list of three-syllable meters? Conjecture how many binary numbers have five digits and list them. (Extra Credit: Learn more about the binary number system and determine whether your answer is correct.)

## 3. Hemachandra-Fibonacci Numbers

The 12th-century writer Ācārya Hemachandra also studied poetic meter. A mora is the durational unit of Sanskrit poetry; short syllables count as one mora and long syllables two morae, which we'll call "beats." Instead of counting meters with a fixed number of syllables, Hemachandra counted meters having a fixed duration. For example, the three meters of three beats are SL, LS, and SSS. More meters are listed in Figure 1.

## Exercises

6. Before you go on, count the number of meters for duration one through five and make a conjecture about the number of meters with six beats and the formula for finding the number of meters with any arbitrary number of beats. Try Worksheet A: Rhythm Patterns of Fixed Duration (p. 17) for a more-thorough exploration of the problem.

| 1 beat | 2 beats | 3 beats | 4 beats |  | 5 beats |  |  |  |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| S | L | SS | SL | LS | LL | SLS | SLL | LLS |
|  |  |  |  | SSS | SSL | LSS | LSL | SSLS |
|  |  |  |  |  |  | SSSS | SSSL | SLSS |
|  |  |  |  |  |  |  |  | LSSS |
|  |  |  |  |  |  |  |  | SSSSS |

Figure 1. Meters listed by duration.

Hemachandra noticed that each number in the sequence is the sum of the two previous numbers. Since the first two numbers are 1 and 2 , the numbers form the sequence $1,2,3,5,8,13, \ldots$. In other words, he discovered the "Fibonacci" numbers-about 50 years before Fibonacci did. Indian poets and drummers know these numbers as Hemachandra numbers. Fibonacci may have learned the sequence from the Indians. Fibonacci was educated in North Africa and was familiar with Eastern mathematics. His book Liber Abaci (1202), in which the sequence appears, introduced the Indian positional number system-the system that we use today-to the West. However, his description of the number sequence as counting the sizes of successive generations of rabbits is not found in India.
Theorem 2. The sequence of numbers of meters with $n$ beats, beginning with $n=1$, is the Hemachandra sequence, $1,2,3,5,8,13, \ldots$. When $n>2$, each number in the sequence is the sum of the two previous numbers.

Proof: Let $H[n]$ represent the $n$th number in the sequence, the total number of patterns of duration $n$. Since there is one pattern (S) of duration 1, we have $H[1]=1$; and since there are two patterns ( SS and L ) of duration 2, we have $H[2]=2$. When $n>2$, partition the collection of $n$-beat patterns into two sets: patterns of duration $(n-2)$ followed by a long syllable L , and patterns of duration $(n-1)$ followed by a short syllable $S$. The number of patterns in the first set is $H[n-2$ ], since they are formed by adding L to the patterns with $(n-2)$ beats; and the number of patterns in the second set is $H[n-1]$, since they are formed by adding $S$ to the patterns with $(n-1)$ beats. The partition shows that $H[n]=H[n-1]+H[n-2]$ when $n>2$. Therefore, the list of numbers forms the Hemachandra sequence. $\square$

If the notation $H[n], H[n-1]$, etc. is unfamiliar, it's worth taking time to understand it. Because $H[n]$ refers to the $n$th number in the sequence, $H[n-1]$ is the $(n-1)$ st number-that is, the number preceding $H[n]$. The equation $H[n]=H[n-1]+H[n-2]$ means "the $n$th number is the sum of the number that is one place before it and the number that is two places before it." For example, if $n=5$, then

$$
H[5]=H[5-1]+H[5-2]=H[4]+H[3] .
$$

In words, the fifth number is the sum of the fourth number and the third number. Note that $H[5-1] \neq H[5]-1$, because $H[5-1]=H[4]=5$, while $H[5]-1=8-1=7$.

## Exercise

7. The procedure for writing the patterns with duration $n$ as a combination of patterns of durations $(n-1)$ and $(n-2)$, as explained in the proof, suggests how to use the information in Figure 1 to list the 13 patterns with six beats. Write them out.

### 3.1 Recursion

Recursion is a process in which one structure is embedded inside another similar structure, rather like nesting Russian dolls, or the Droste cocoa box. Recursion is the lifeblood of computer programming and is crucial in mathematics as well. An algorithm is a recursive if you start with some information (called a base case) and arrive at all subsequent information by repeatedly applying the same rule, called a recursive rule.

In the example of the Hemachandra numbers, there are two meters of one syllable each, L and S . This is the base case. If you know all the meters that have $n-1$ or $n-2$ syllables, you can list the meters that have $n$ syllables by first adding an L to the beginning of the meters with $n-2$ syllables, then adding an S to the beginning of the meters with $n-1$ syllables. This is the recursive rule that is expressed by the formula $H[n]=$ $H[n-1]+H[n-2]$.

### 3.2 The Padovan Sequence

The poetic meters that Pingala and Hemachandra studied have an analogue in music. Music from India, the Middle East, and the Balkans is often written in additive meter-that is, a rhythmic organization founded in grouping beats rather than subdividing larger units of time called measures, which is the typical structure of Western European music.

Figure 2 shows a few examples. The Bulgarian dance called Daichovo horo has a nine-beat measure, grouped $2+2+2+3$. This means that the first, third, fifth, and seventh beats normally receive an accent; they are also the beats on which the dancers step. The jazz pianist and composer Dave Brubeck (1920-2012) used the same rhythm in his "Blue Rondo à la Turk" (1959). A Gankino horo has an eleven-beat measure, with beats grouped $2+2+3+2+2$.

Many additive meters are binary codes formed of two- and three-beat groupings, rather than one- and two-beat groupings. In this situation, we need something like Hemachandra's sequence for counting meters of a given duration, as explored in the following exercise.

Rhythms of one- and two-beat notes merengue bell part (Dominican Rep.)
cumbia bell part (Columbia)
mambo bell part (Cuba)
bintin bell pattern (Ghana)
also bembe shango (Afro-Cuban)
Rhythms of two- and three-beat notes
lesnoto (Bulgaria)
bomba (Puerto Rico)
guajira (Spain)
12-beat clave (Cuba)


Figure 2. Musical rhythms from various cultures.

## Exercise

8. This problem is explored in Worksheet A: Rhythm Patterns of Fixed Duration (p. 17). The Hemachandra numbers count rhythms formed from one- and two-beat notes. What sequence counts rhythms consisting of two- and three-beat notes? Find the first few numbers in the sequence-the base case-and a recursive rule that generates the sequence. Explain why your rule is correct. This number sequence is called the Padovan sequence and has a rich history. The sequence is named after architect Richard Padovan (1935- ), despite Padovan's specific attribution of the sequence to the Dutch architect (and later monk) Hans van der Laan (1904-1991), who investigated proportions in architecture.

The On-Line Encyclopedia of Integer Sequences [Sloane 2020] is a wonderful research tool for investigating sequences and their history.

## 4. The Expanding Mountain of Jewels

Pingala is credited with the discovery of "Pascal's" triangle in India, which he called the meruprastāra, or "the expanding mountain of jewels," referring to the mythical Mount Meru made of gold and precious stones. The $n$th row in this triangle counts the number of unordered combinations of $n$ syllables considered all different (rather than just long or short): taking all $n$ syllables together, taking all combinations of $(n-1)$ syllables, taking all combinations of $(n-2)$ syllables, and so on.

Here is how to compute the third row in the meruprastāra. We have:
1 way to choose three syllables from "prastāra" (that is, pras + tā + ra),
3 ways to choose two syllables (pras + tā, pras + ra, tā + ra),
3 ways to choose one syllable (pras, tā, ra), and
1 way to choose no syllables.

Therefore, the third row is 1331 .
When each list counting combinations of $r$ syllables drawn from sets of $n$ syllables is written in a row, and the rows are stacked, the numbers form a triangular array that extends forever:

|  |  |  | 1 |  | 1 |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  | 1 |  | 2 |  | 1 |  |  |
|  | 1 |  | 3 |  | 3 |  | 1 |  |
| 1 |  | 4 |  | 6 |  | 4 |  | 1 |

Pingala recognized that each interior number is the sum of the two numbers above it. This array is known in the West as Pascal's triangle-though, of course, it wasn't yet named for Pascal, who was born in France in 1623. This triangle was long recognized all over the world previously. Figure 3 shows prior images from three cultures.


Figure 3. The meruprastāra from North Africa (c. 1150) to China (1303) to Germany (1527).

## Exercise

9. Kedara Bhatta (14th century) discovered the meruprastāra in a different context: He found the number of meters of $n$ syllables having $r$ short syllables. This is the problem that is solved in Worksheet A. The fact that the two problems produce the same triangle is, of course, no coincidence. Find an exact correspondence between the number of combinations of $r$ objects drawn from a set of $n$ different objects and the number of meters of $n$ syllables having $r$ short syllables.

### 4.1 Recursion and the Meruprastāra

The first row of the meruprastāra contains the numbers 1, 1. This is the base case. If you know any row in the meruprastāra, each number in the following row equals the number directly above it plus the number
diagonally above and to the left (if there is no number in these positions, add zero). This is the recursive rule.

The 12th-century writer Bhaskara gives another recursive algorithm for finding the numbers in the meruprastāra in his work Lilavati. To find the $n$th row in the meruprastāra, start by writing the numbers counting up $(1,2, \ldots, n)$, and then above them write the numbers counting down ( $n, n-1, \ldots, 2,1$ ), as we show for $n=5$ :

54321
12345
The first number for the row of the meruprastāra is 1 (this is true for every $n$ ). Obtain the other numbers in the row by successively multiplying and dividing by the numbers that you have written:

$$
\frac{5}{1}=5 ; \quad 5 \cdot \frac{4}{2}=10 ; \quad 10 \cdot \frac{3}{3}=10 ; \quad 10 \cdot \frac{2}{4}=5 ; \quad 5 \cdot \frac{1}{5}=1 .
$$

This tells us that the fifth row is

$$
15101051
$$

The numbers in row $n$ are built up recursively, one from the next, starting from 1 (the base case).

## Exercise

10. Find the sixth row of the meruprastāra using Bhaskara's method. Check your work using the addition algorithm, starting with the fifth row.

## 5. Naming Meters and de Bruijn Sequences

### 5.1 Remembering Meters

Since there are hundreds of Sanskrit meters, remembering the pattern for any particular meter requires some effort. Pingala's indexing procedure helps somewhat; for example, it identifies the pattern LLLSLS as "number 41 in the catalog of six-syllable meters." However, Pingala's best and most well-known solution to the problem of remembering meters involves the following mapping of groups of three syllables to letters:

| $m$ | LLL | $r$ | LSL | $t$ | LLS | $b h$ | LSS |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $y$ | SLL | $s$ | SSL | $j$ | SLS | $n$ | SSS |

To encode the meter LLLSLS, begin by breaking it into groups of threes (LLL-SLS). These groups correspond to the letters $m$ and $j$. The letters $m j$
can be embedded in a word-say, "mojo"-that is more memorable that "number 41." Musicians too use this method for remembering rhythm patterns.

Either in Pingala's time or later, the nonsense word yamātārājabhānasalagām
came to be used as a way to remember the mapping of triplets of syllables to letters. The word defines a pattern of long and short syllables (in the English transliteration of Sanskrit, $a$ is a short vowel and $\bar{a}$ is a long vowel):

| $y a$ | $m \bar{a}$ | $t \bar{a}$ | $r \bar{a}$ | $j a$ | $b h \bar{a}$ | $n a$ | $s a$ | $l a$ | $g \bar{a} m$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| S | L | L | L | S | L | S | S | S | L |

The pattern SLLLSLSSSL has the curious property that each of the eight sequences of three syllables occurs exactly once. For example, the first three syllables form the pattern SLL, the second three syllables form LLL, and so on. These patterns are named using Pingala's table, so that ya represents SLL. The number of syllables in a meter does not have to be a multiple of three, the last two syllables, and la and gām are used to fill out the last pattern. It is not known whether Pingala knew this mnemonic for the triplets, or if it was discovered by poets and drummers who came after him.

## 5.2 de Bruijn Sequences

The mathematician Sherman Stein (1926-) recognized that the pattern SLLLSLSSSL is close to being a de Bruijn sequence. A de Bruijn sequence is a binary code where, if the pattern were wrapped on a circle, every "word" of $n$ letters would appear exactly once. The pattern SLLLSLSS-the pattern above minus the last L-is a binary de Bruijn sequence for three-letter words. Figure 4 depicts de Bruijn sequences for three- and four-letter words arranged on circles.


Figure 4. De Bruijn sequences for binary codes of three and four letters, arranged on circles.

## Exercise

11. Although LLLSLSSS is also a de Bruijn sequence, it is not fundamentally different from SLLLSLSS, since each sequence produces the three-letter words in the same order, though starting at a different point in the cycle. Using this notion of equivalence, there are only two possible de Bruijn sequences for three-letter words that use the alphabet $\mathrm{L}, \mathrm{S}$. How can we find the other one?

## 6. Patterns in Music and Architecture

Indian scholars had a great enthusiasm for solving mathematical problems related to poetry. However, the situation of pattern in Indian music tells a more remarkable story. There, the list of patterns has transcended its role as a "dictionary" of available patterns and has become itself an interesting and valuable musical structure. In music, prastāra-meaning systematic permutation of rhythmic elements-is commonly recognized as a principal part of the process of rhythmic variation. Indian musicians typically use prastāra towards the end of a piece, since progression through all the permutations of a rhythmic pattern is a process that has a definite ending-that is, when all the possibilities have been exhausted. Lewis Rowell's description of prastāra in early Indian music is also applicable today:

Once again we can draw an important formal conclusion from the popularity of prastāra: endings are to be signaled well in advance by the onset of some systematic musical process, a process of playful exploitation that can be followed along a course of progressively narrowed and focused expectations and that leads inexorably to a predictable conclusion. ... But prastāra has symbolic overtones that transcend its local role as a simple tactic of closure: the device mimics the series of transformations through which all substance must eventually pass.

Rowell [1992, 251]
Figure 5 demonstrates the use of permutation in a simple composition for tabla (Indian drums). Each variation on the theme is a permutation of groups of $4,6,8$, and 10 beats:

| I. | II. | III. | IV. | V. | VI. | VII. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 68666 | 664664 | 646646 | 10610 (6 | 4) 61066 | 610610 | 66686 |

Variations II and III state two of the three permutations of $\{6,6,4\}$, with $4+6+6$ missing. Variations IV, V, and VI exhaust the permutations of the 10and 6-beat phrases (note that variation V begins with the last four beats of the 10-beat phrase and combines with the last 6 beats of variation VI to form the 10-beat phrase). Finally, variation VII, a mirror image of I, signals that the permutation process has come to a close.

| Theme | Dhi <br> ta <br> ta <br> ghi | ne <br> ke <br> ge <br> de | Ta <br> ta <br> Dhi <br> na | ge <br> ge <br> ne <br> ge | Dhi <br> Dhi <br> Ta <br> Tin | ne <br> ne <br> ge <br> na | Ta <br> Ta <br> te <br> Ta | ge <br> ke <br> te <br> ke |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Variation I. | (ta | ge | Dhi | ne | Ta | ke) | (ta | ge |
| $6+8+6+6+6$ | Dhi | ne | Ta | ke | ta | ke) | (ta | ge |
|  | Dhi | ne | Ta | ke) | (ta | ge | Dhi | ne |
|  | Ta | ke) | (ta | ge | Tin | ne | Ta | ke) |
| Variation II. | \|: (ta | ge | Dhi | ne | Ta | ke) | (ta | ge |
| $6+6+4+6+6+4$ | Dhi | ne | Ta | ge) | (Dhi | ne | ta | ke) :\| |
| Variation III. | \|: (ta | ge | Dhi | ne | Ta | ge) | (Dhi | ne |
| $6+4+6+6+4+6$ | Ta | ke) | (ta | ge | Dhi | ne | ta | ke) :\| |
| Variation IV. | \|: (ta | ge | Dhi | ne | Ta | ke | ta | ke |
| $10+6+10+6$ | ta | ke) | (ta | ge | Dhi | ne | Ta | ke) : \| |
| Variation V. | \| ta | ke | ta | ke) | (ta | ge | Dhi | ne |
| $4+6+10+6+6$ | Ta | ke) | (ta | ge | Dhi | ne | Ta | ke :\| |
| Variation VI. | \|: (ta | ge | Dhi | ne | Ta | ke) | (ta | ge |
| $6+10+6+10$ | Dhi | ne | Ta | ke | ta | ke | ta | ke) :\| |
| Variation VII. | (ta | ge | Dhi | ne | Ta | ke) | (ta | ge |
| $6+6+6+8+6$ | Dhi | ne | Ta | ke) | (ta | ge | Dhi | ne |
|  | Ta | ke) | (ta | ge | Dhi | ne | Ta | ke |
|  | ta | ke) | (ta | ge | Dhi | ne | Ta | ke) |

Figure 5. Theme and variations for tabla, as taught by Lenny Seidman. Syllables such as "Dhi" and "ne" indicate particular ways of hitting the drums. Each syllable occupies the same amount of time. The symbols $\mid:$ and :|indicate repeats, and parentheses enclose phrases.

As Rowell [1992] points out, we can also understand prastāra as manifesting a fascination with recursive generation and transformation that appears in Indian art, architecture, and religion from ancient times. The medieval Śekharī ("multi-spired") temples of western and central India gave form to the view that the cosmos was recursively generated. The eleventhcentury Kandāriyā Mahādeva temple (Figure 6) is a celebrated example of this style; it is composed of miniature shrines (aedicules) emanating from a central shrine. Concerning the structure of such temples, Adam Hardy writes:

As soon as the dynamic relationships between the aedicules are considered, the vision of a theological hierarchy can be seen as a dynamic process of manifestation: the emerging, expanding, proliferating, fragmenting, dissolving patterns are so closely analogous to the concept, perennial in India, of a world of multiplicity recurrently manifesting from unity and dissolving back into unity, that the idea can be said to be embodied in the forms. Hardy [2002, 91-92]
Figure 7 shows how the meruprastāra is made of copies of itself. It is no wonder that ancient and medieval Indian mathematicians developed an outstanding facility with recursion.


Figure 6. Recursion in Indian architecture: the 11th-century Kandāriyā Mahādeva temple [Thérond 1876, 340].


Figure 7. The meruprastāra is made of copies of itself.

## 7. Solutions to the Exercises

1. There are several algorithms that may be used. One is to write all the 4 -syllable meters, following each with an L, then write the 4 -syllable meters again, following each with an S.

| LLLLL | SLLLL | LSLLL | SSLLL | LLSLL | SLSLL | LSSLL | SSSLL |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| LLLSL | SLLSL | LSLSL | SSLSL | LLSSL | SLSSL | LSSSL | SSSSL |
| LLLLS | SLLLS | LSLLS | SSLLS | LLSLS | SLSLS | LSSLS | SSSLS |
| LLLSS | SLLSS | LSLSS | SSLSS | LLSSS | SLSSS | LSSSS | SSSSS |

2. Since LLSSLSL has seven syllables, the list has $2^{7}=128$ meters.
3. Pingala's algorithm demonstrates that each list is twice the size of the previous one (formally, if there are $k$ meters of $(n-1)$ syllables, there are $2 k$ meters of $n$ syllables. Since there are 2 one-syllable patterns, the numbers of meters of each length follows the pattern $2,4,8,16$, and so on.
4. Since patterns of H and T are binary, there are $2^{3}=8$ patterns, and they are HHH, THH, HTH, TTH, HHT, THT, HTT, TTT. Each of these patterns is equally likely, and three of them have one T and two H's, so the likelihood of getting one T is $3 / 8$ or $37.5 \%$. In general, use the algorithms from the study of meters, substituting H for L and T for S .
5. Start with 1 , then use the list of three-syllable patterns, replacing 0 with $\mathrm{L}, 1$ with S, and writing the patterns backwards. There are $2^{4}=16$ fourdigit binary numbers. You write them by following this same procedure with the list of three-syllable meters. Proving that this answer is correct involves understanding how place-value number systems work in bases other than 10.
6. If your conjecture was something like "there are 13 meters with 6 beats, and you get any number in the sequence by adding the two previous," you would be correct.
7. The procedure is to add an $L$ to all the four-beat patterns, then add an $S$ to all the five-beat patterns, which are listed in Figure 1. The answer is:
```
LLL SSLL SLSL LSSL SSSSL
SLLS LSLS SSSLS LLSS LLSLS SLSSS LSSSS SSSSSS
```

8. Here are the first ten entries of the Padovan sequence:

$$
\begin{array}{c|lllllllllr}
\text { duration } & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text { patterns } & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 7
\end{array}
$$

If we let $P[n]$ be the number of $n$-beat rhythms, then a recursive rule is

$$
P[n]=P[n-2]+P[n-3]
$$

when $n>3$. The proof of this statement is similar to the proof of Theorem 2 (the Hemachandra numbers). In this case, partition the rhythms of duration $n$ into rhythms of duration $(n-2)$ followed by a two-beat note and rhythms of duration $(n-3)$ followed by a three-beat note.

Incidentally, a student noticed that

$$
P[n]=P[n-1]+P[n-5]
$$

holds for several values of $n>5$ and conjectured that all Padovan numbers follow this rule. Is this correct? Hint: Use the first rule to rewrite $P[n-1]$.

The Padovan sequence has some beautiful properties-for example, it is related to a spiral of equilateral triangles in the way that the Hema-chandra-Fibonacci sequence is related to a spiral of squares (see Figure 9); and it is closely connected to the Perrin sequence, which uses the same recursive relationship as the Padovan sequence but with different starting values. See Stewart [2004, 85-94] for more examples.


Figure 9. A spiral of squares from the Hemachandra-Fibonacci sequence, and a spiral of triangles from the Padovan sequence.
9. Any way of choosing objects from a collection of $n$ different objects can be represented by a binary code of length $n$ in this way: Assign numbers from 1 to $n$ to the objects, and write I (in) if an object is selected and O (out) if it is not. For example, suppose that you choose $\{2,5,7\}$ from the collection $\{1,2, \ldots, 8\}$. That choice corresponds to the binary code OIOOIOIO. Each choice of $r$ objects corresponds to a pattern of $n$ letters with $r$ I's. Substituting $S$ for I and L for O produces an $n$-syllable meter with $r$ short syllables.
10. 654321

123456
The first number in the row is 1 (this is true for every $n$ ). Obtain the other numbers in the row by successively multiplying and dividing by the numbers that you have written:

$$
\frac{6}{1}=6 ; 6 \cdot \frac{5}{2}=15 ; 15 \cdot \frac{4}{3}=20 ; 20 \cdot \frac{3}{4}=15 ; 15 \cdot \frac{2}{5}=6 ; 6 \cdot \frac{1}{6}=1 .
$$

This tells us that the sixth row is 1615201561 .
11. Consider the eight three-letter words. A de Bruijn sequence can be thought of as an ordering of these words: The first word is the first three letters in the sequence; the second word is the second through fourth letters, and so on. You probably have noticed that there are some rules about which words can follow each other. For example, SSL can be followed by either SLS or SLL. A powerful representation called a directed graph is useful in organizing these possibilities. The vertices of the graph represent states (in this case, three-letter words). An arrow between two states indicates that the first state can be followed by the second. The graph is shown in Figure 10. Any path that visits each vertex exactly once defines a de Bruijn sequence. The yamātārājabhānasalagām sequence comes from following the path shown in red and also to the right of the directed graph.


Figure 10. Directed graph showing which three-letter words can follow each other.

## 8. Worksheets

### 8.1 Worksheet A: Rhythm Patterns of Fixed Duration

Musical patterns are sometimes classified by their number of beats, called their duration. Many patterns use one-beat notes and two-beat notes, often called quarter notes ( $($.$) and half notes ( (\mathrm{)}$. We can represent them by " 1 " and "2."

$$
\text { duration(1121) }=1+1+2+1=5
$$

If we know how many beats we want to fill (that is, the total duration), how many ways are there to do it, using only one-beat and two-beat notes? We can figure out the answer by creating a list:

| Duration | Patterns | Number of patterns |
| :---: | :--- | :--- |
| 1 | 1 | 1 |
| 2 | 11,2 | 2 |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |

1. Complete the table.
2. How many patterns have duration 7? Duration 8? Duration 9?
3. Explain how you find the next number in the list of numbers of patterns.
4. What is the Western name for this sequence of numbers?

Relationship to poetry. Syllables in Sanskrit poetry (and in many other languages) are either short or long. The duration of a long syllable is exactly twice that of a short syllable. This means that we can measure the total duration of a line of poetry, using " S " for a short note and " L " for a long note. For example, the duration of SSLS is 5. The Indian scholar Hemachandra (c. 1100 A.D.) investigated the following question: How many poetic meters are there of any given duration? Use what you have discovered in the exercise above to answer his question.

### 8.2 Worksheet B: Musical Rhythms with 2- and 3-Beat Notes

In this worksheet, you will derive the sequence counting meters of a given duration formed from notes of length 2 and 3 .

| Duration | Patterns | Number of patterns |
| :---: | :--- | :--- |
| 1 | (none) | 0 |
| 2 | 2 | 1 |
| 3 | 3 | 1 |
| 4 | 22 | 1 |
| 5 | 23,32 | 2 |
| 6 | 33,222 | 2 |
| 7 | $223,232,322$ | 3 |
| 8 |  |  |
| 9 |  |  |

1. Complete the table.
2. Look for a pattern in the right-hand column. How many rhythms have duration 10? Duration 11?
3. Explain how you find the next number in the sequence.
4. Suppose that one-, two-, and three-beat notes are allowed. How many rhythms have a given duration?
5. Many rhythms in Western music are formed of whole notes, half notes, and quarter notes. How many rhythms have duration equal to $n$ whole notes?
6. Suppose that notes of duration one through $n$ are allowed. How many rhythms have total duration $n$ ? The answer is surprisingly familiar. Can you explain it in a different way?

### 8.3 Worksheet C: The Expanding Mountain of Jewels

In each box below, enter the number of meters that fit the description. The number in the right-hand column should equal the sum of the numbers in that row. This pattern was first noticed by Pingala when solving a slightly different problem. He called this pattern the meruprastāra, or "Expanding Mountain of Jewels."


1. Describe an algorithm that tells you how the find the $n$th row in the table, if you know the previous row.
2. What is the name for this array in Western mathematics?
3. What are other things that the numbers in this array count?
4. How is this array related to the Hemachandra numbers?

### 8.4 Worksheet D: Patterns in the Meruprastāra

There are some other cool recursive patterns in the meruprastāra. In the figure below, shade the odd numbers and leave the even numbers white.


In fact, you can predict which numbers will be odd and which numbers will be even, just by knowing how odd and even numbers add:

$$
\begin{aligned}
\text { odd }+ \text { odd } & = \\
\text { odd }+ \text { even } & = \\
\text { even }+ \text { even } & =
\end{aligned}
$$

So you don't need to calculate the numbers to shade in the hexagons!
Make a patterned meruprastāra using hexagonal graph paper with $1 / 4^{\prime \prime}$ spacing, which you can print at
https://www.printablepaper.net/category/hexagon_graph

Turn the paper sideways. Start by coloring the hexagons that would contain 1's, then color the others using the addition rules for odd and even numbers.

Research Sierpiński's triangle. How is this picture related? What is a fractal?

## 9. Quizzes

### 9.1 Quiz 1

Suppose that poetic meters can be made of syllables of length $S$ ( 1 beat), $\mathrm{L}(2$ beats), and E (3 beats). Let $T[n]$ be the number of meters of total duration $n$ beats.

1. Find $T[1], T[2]$, and $T[3]$ by listing the patterns.
2. If, for all $n>3, T[n]=T[n-1]+T[n-2]+T[n-3]$, then
(a) Fill in the blanks: $T[4]=T[\quad]+T[\quad]+T[\quad]$.
(b) Use the rule and your answers to the previous question to find $T[4]$, $T[5]$, and $T[6]$. Check your work by finding the patterns that are counted by $T[4]$.

### 9.2 Quiz 2

1. The fifth row in Pascal's Triangle is 15101051.
(a) What is the sixth row?
(b) What does the 10 in the fifth row count, as related to Sanskrit poetry?
(c) How many binary codes formed of 0's and 1's have one 0 and four 1's?
2. How many binary codes have seven letters?

## 10. Solutions to Quizzes

### 10.1 Solutions to Quiz 1

1. $T[1]=1$ because the only meter is S .
$T[2]=2$ because the meters are SS and L .
$T[3]=4$ because the meters are SSS, SL, LS, and E.
(a) $T[4]=T[4-1]+T[4-2]+T[4-3]=T[3]+T[2]+T[1]=$ $1+2+4=7$. The seven patterns counted by $T[4]$ are SE, SSL, LL, SSSS, SLS, LSS, and ES.
(b) $T[4]=T[3]+T[2]+T[1]=1+2+4=7$.
$T[5]=T[4]+T[3]+T[2]=7+4+2=13$.
$T[6]=T[5]+T[4]+T[3]=13+7+4=24$.

### 10.2 Solutions to Quiz 2

1. (a) The sixth row is 1615201561 .
(b) The 10 in the fifth row counts the number of patterns that have five letters, where three are $S$ and two are $L$ (equivalently, two are $S$ and three are L).
(c) There are 5.
2. There are $2^{7}=128$ codes with 7 letters.

## 11. Sample Exam Questions

1. Suppose that you flip a coin four times and write the outcome as a sequence of heads and tails, like HTHH. How many different sequences of four flips are possible? List them.
2. For each sequence of heads and tails in the previous exercise, count the number of T's and H's. For example, HTHH has 3 H's and 1 T. Of the patterns in the previous exercise, how many have 4 T's? 3 T's? 2 T's? 1 T? no T's? Which row of the Pascal triangle corresponds with your answers?
3. Find the number of binary codes of length 10 made from 0 and 1 . Of those, how many start with a 0 ? How many start with a 1? Explain.
4. Find the number of ways 1-beat notes (1's) and 2-beat notes ( 2 's) form a pattern of duration 9 beats. Of those patterns, how many start with a 1? How many start with a 2? Explain how you know.
5. Consider the collection of poetic meters with seven syllables, which may be long or short. How many total meters are there? How many of these meters have
a) no short syllables?
b) one short syllable?
c) two short syllables?
d) three short syllables?
e) four short syllables?
f) five short syllables?
g) six short syllables?
h) seven short syllables?

Hint: This question is extremely tedious if you try to write down all the meters and count them. The best way to answer it is to use the relationship between poetic meters and the meruprastāra (Pascal's triangle).
6. Suppose that you flip a coin seven times and write down the sequence of heads H and tails T. There is an exact correspondence between sequences of heads and tails and poetic meters: Just replace H with L and T with S. How many possible patterns are there? Use your answers to the previous question to determine the percentage of the patterns, rounding to two decimal places, that have
a) no tails.
b) one tail.
c) two tails.
d) three tails.
e) four tails.
f) five tails.
g) six tails.
h) seven tails.

Assuming that your coin is fair, each of the patterns of heads and tails is equally likely. The percentages are the probabilities of each number of tails when you flip the coin seven times.
7. How many meters have the same duration as SSSSSS (including SSSSSS)? How have the same duration as LLLLLL? Your answers should be numbers in the Hemachandra sequence.
8. Suppose that a drummer wants to take a solo that is eight beats long and made up of 1-beat notes, 2-beat notes, and 4 -beat notes. How many patterns are possible? Hint: Don't try to list the patterns, because there are a lot. Rather, find base cases and a recursive rule, write the sequence of number of patterns, and find the eighth number in that sequence. If you're a musician, the question is, "How many ways can you fill two measures in $4 / 4$ time with quarter notes, half notes, and whole notes?"
9. Add the numbers between the diagonal lines in Pascal's triangle. For example, the first few sums are $1,1,1+1=2,1+2=3$, and $1+3+1=5$. What's the pattern? See Figure 12.


Figure 12.
10. Suppose that a sequence of numbers $S[1], S[2], S[3], \ldots$ is defined recursively by $S[n]=S[n-1]+S[n-3]$.
a) If the first four numbers in the sequence are $1,1,2,3$, what is the fifth number?
b) Fill in the blanks: $S[100]=S[\quad]+S[\quad]$.
11. Suppose that poetic meters are made of short syllables $S$ worth 1 beat, medium syllables $M$ worth 2 beats, and long syllables $L$ worth 3 beats. Write all the meters that have total duration equal to 4 beats. For extra credit, write a recursive rule for this type of pattern.
12. Find the sixth row of the meruprastāra, starting with the fifth row, which is 15101051 .
13. Suppose that poetic meters are made of medium syllables M worth 2 beats and long syllables $L$ worth 3 beats. Write all the meters that have total duration equal to 8 beats.

## 12. Solutions to Sample Exam

1. There are $2^{4}=16$ sequences, because the sequences are binary codes. They are HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT.
2. There is 1 pattern with 4 T's, 4 patterns with 3 T's, 6 patterns with 2 T's, 4 patterns with 1 T , and 1 pattern with no T's. The numbers $1,4,6,4,1$ form the fourth row in Pascal's triangle.
3. There are $2^{10}=1024$ codes. Because the length- 10 codes are found by adding a 1 to the end of each of the length- 9 codes, then adding a 0 to the end, exactly half (512) begin with 0 and half begin with 1.
4. Patterns with duration 9 beats are formed by either adding a 1 to the beginning of an 8 -beat pattern, or adding a 2 to the beginning of a 7 beat pattern. There are 348 -beat patterns and 217 -beat patterns.
5. There are $2^{7}=128$ meters. The numbers of meters of each type are found in row 7 of the meruprastāra: a) 1 , b) 7 , c) 21 , d) 35 , e) 35 , f) 21 , g) $7, \mathrm{~h}) 1$.
6. There are $2^{7}=128$ patterns. The percentages of patterns of each type are found by dividing row 7 of the meruprastāra by 128: a) $0.78 \%$, b) $5.47 \%$, c) $16.41 \%$, d) $\%$, e) $27.34 \%$, f) $16.41 \%$, g) $5.47 \%$, h) $0.78 \%$.
7. SSSSSS has duration $1+1+1+1+1+1=6$, so the answer is the 6 th Hemachandra number: 13. LLLLLL has duration 12 , so the answer is the 12th Hemachandra number: 233.
8. The technique is to follow the procedure in Worksheet B. There are

- 1 pattern with duration 1 beat: 1
- 2 patterns with duration 2 beats: $1+1,2$
- 3 patterns with duration 3 beats: $1+2,1+1+1,2+1$
- 6 patterns with duration 4 beats: $4,1+1+2,2+2,1+2+1,1+1+1+1,2+1+1$

These comprise the base case. The recursive rule is that each number in the sequence is the sum of the numbers that are one, two, and four places before it. In mathematical notation, if we let $D[n]$ be the number of patterns with $n$ beats, then $D[n]=D[n-1]+D[n-2]+D[n-4]$. The proof of this is similar to the proof of Theorem 2. Using the recursive rule, the sequence is $1,2,3,6,10,18,31,55, \ldots$, so the answer is 55 , the eighth number.
9. The answers are Hemachandra/Fibonacci numbers: $1,1,2,3,5,8, \ldots$.
10. a) The rule says that $S[5]=S[5-1]+S[5-3]=S[4]+S[2]=4$.
b) $S[100]=S[99]+S[97]$.
11. SL, SSM, MM, SSSS, SMS, MSS, LS.
12. 161520156 1. Adding adjacent numbers in the fifth row gives the answer.
13. MMMM, LLM, LML, MLL

## References

Hall, Rachel. 2020. Math for drummers. UMAP Modules in Undergraduate Mathematics and Its Applications: Module 809. The UMAP Journal of Mathematics and Its Applications 41 (1): 35-60.
Hardy, Adam. 2002. Śehkarī temples. Artibus Asiae 62 (1): 81-137.
Rowell, Lewis. 1992. Music and Musical Thought in Early India. Chicago, IL: University of Chicago Press.
Sloane, N.J.A. (ed.). 2020. The On-Line Encyclopedia of Integer Sequences. https://oeis.org.
Stewart, Ian. 2004. Tales of a neglected number. Chapter 8 in Math Hysteria: Fun and Games with Mathematics, 85-93. New York: Oxford University Press.

Thérond, Émile. 1876. The temple of Mahadeva, Kajraha. In Rousselet, Louis, India and Its Native Princes: Travels in Central India and in the Presidencies of Bombay and Bengal, 340. New York: Scribner, Armstrong, and Co.

## Acknowledgments

This UMAP Module is adapted from a chapter in Dr. Hall's forthcoming book The Sound of Numbers, which in turn has arisen from her long-time teaching of a liberal arts mathematics course in mathematics and music.

## About the Author

Rachel Hall is an associate professor of mathematics at Saint Joseph's University, where she researches and teaches both mathematics and music. She is writing a book entitled The Sound of Numbers. She is an avid shape-note singer and is a co-author of The Shenandoah Harmony. She plays English concertina, accordion, and piano, and recorded and toured with the folk group Simple Gifts. Another UMAP Module by her, "Math for drummers," has appeared in this Journal (see the References).

