The spectral measure of a dynamical system

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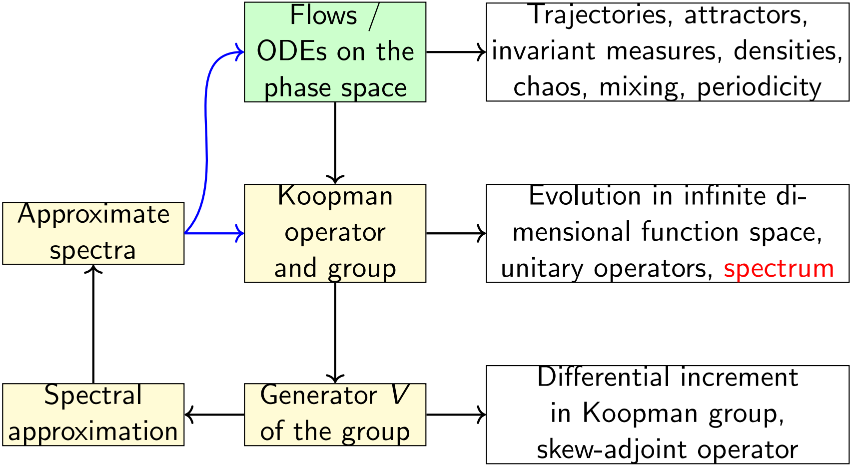
SIAM Minisymposium on Advances in Manifold Learning and Applications,
Joint Mathematics Meetings (JMM21)
January 09, 2021

joint work with Dimitris Giannakis, Joanna Slawinska

Funded by NSF, ONR and DARPA.

Outline

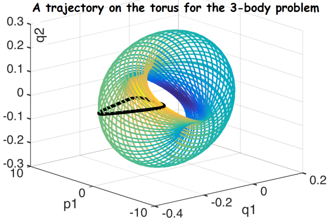

Flows / ODEs on the phase space
Trajectories, attractors, invariant measures, densities, chaos, mixing, periodicity
Approximate spectra
Koopman operator and group
Evolution in infinite dimensional function space, unitary operators, spectrum
Spectral approximation
Generator V of the group
Differential increment in Koopman group, skew-adjoint operator

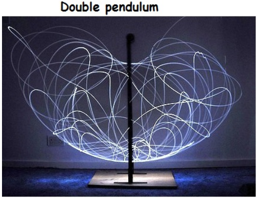
Continuous-time dynamical systems

Many deterministic physical systems are governed by an ODE

$$dx/dt = V(x).$$

- These equations are defined in a phase-space M, V is some vector field, x is some point in space.
- $\Phi^t x_0$ denotes the unique solution curve to the above ODE with initial point x_0.
- More generally, one could define a flow $\Phi^t : M \to M$, without assuming an underlying vector field.





The Koopman operator U^t

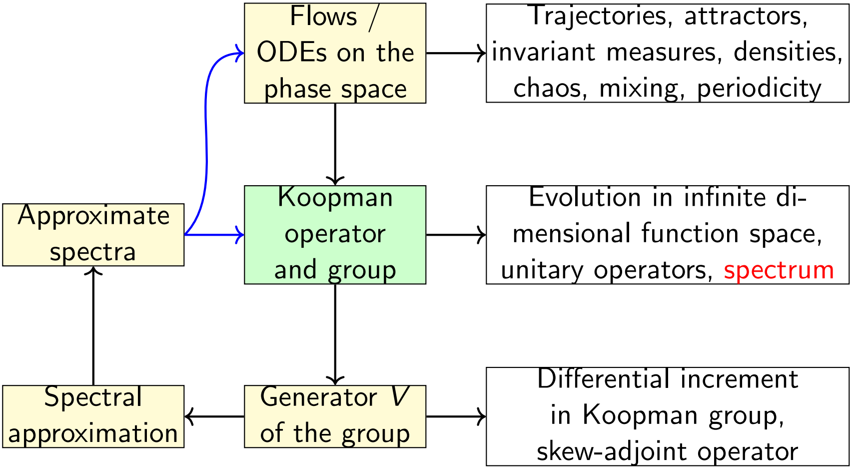
Dynamics / trajectories on phase space $\leftrightarrow$ Dynamics in observation spaces (e.g. $L^2(X,\mu)$ or $C^0(X)$). Given a function f.

$$\left(U^t f\right) \text{ is another observable } : x \mapsto f\left(\Phi^t x\right).$$

$$\begin{array}{ccc} x & \xrightarrow{\Phi^t} & \Phi^t x \\ \downarrow f & & \downarrow f \\ f(x) & \xrightarrow{U^t} & f(\Phi^t x) \end{array}$$

The operator theoretic approach to dynamical systems $\rightarrow$ spectral theory of dynamical systems.

Outline II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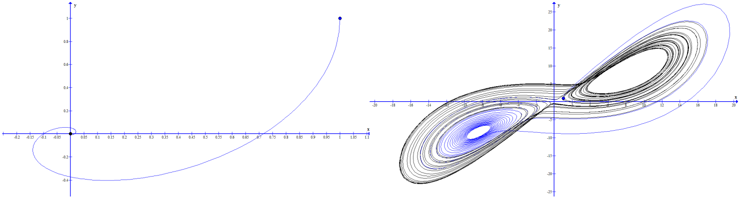
Invariant measure μ and $L^2(\mu)$



- An invariant measure μ on the attractor / invariant set X.
- It is said to be ergodic if there are no proper invariant subsets.
- Ergodicity is closely related to the concept of *equidistribution*, i.e., the statistical properties along a trajectory is the SAME as the statistical properties over the entire phase space.
- The Hilbert space $L^2(\mu)$ of square-integrable functions.

$$\|f\|^2_{L^2(\mu)} = \int_M |f(x)|^2 d\mu(x), \quad \langle f, g\rangle_{L^2(\mu)} = \int_M f(x)\overline{g}(x) d\mu(x)$$

Discrete and continuous components

Koopman eigenfunction : $U^t z = e^{\iota\omega t} z$.

Let $\mathcal{D}$ be the closed subspace spanned by all eigenfunctions. It is non-empty.

$$U^t \circlearrowright \mathcal{D} \oplus \mathcal{D}^\perp \circlearrowleft U^t$$

$\mathcal{D}$ corresponds to quasiperiodic dynamics

$\mathcal{D}^\perp$ is the functional analog of chaotic dynamics. It corresponds to decay of correlations and weak mixing.

Koopman eigenfunctions and forecasting

$$U^t z = e^{\iota \omega t} z$$

U^t is unitary. So all the eigenvalues are **unit-norm**.

- Thus Koopman eigenfunctions have nice *time-evolution*.

$$z\left(\Phi^t x\right) = e^{\iota \omega t} z(x), \quad \forall t, \quad \forall x.$$

- This makes them a useful basis for forecasting.

$$f = \sum_j a_j z_j \Rightarrow U^t f = \sum_j a_j e^{\iota \omega_j t} z_j.$$

- Any collection of eigenfunctions $z_1, \ldots, z_d$ gives an embedded / factor dynamics on the d-torus.

$$\begin{tikzcd} X \arrow[r, "\Phi^t"] \arrow[d, "(z_1,\ldots,z_d)"'] & X \arrow[d, "(z_1,\ldots,z_d)"] \\ \mathbb{T}^d \arrow[r, "R^t_{\vec{\omega}}"] & \mathbb{T}^d \end{tikzcd} ; \qquad R^t_{\vec{\omega}} : \vec{\theta} \mapsto \vec{\theta} + \vec{\omega} \bmod 1$$

Fourier and ergodic averaging

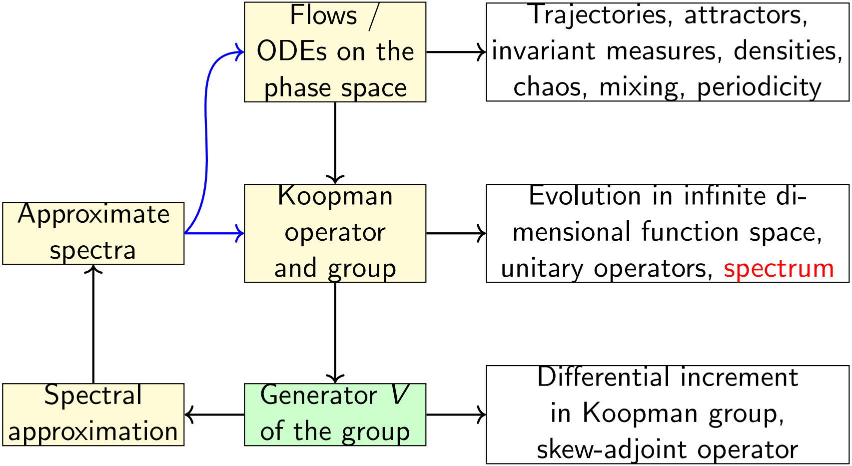
$$\mathcal{F}_{\omega,N}\phi = \frac{1}{N}\sum_{n=0}^{N-1} e^{\iota\omega n} U^{n\delta t}\phi$$

$$\boxed{\begin{gathered}\left\|\mathcal{F}_{\omega,N}\phi\right\|_{L^2(\mu)}^2 = \int_0^{2\pi} \mathcal{S}_N^2(\theta-\omega)\,d\mu_\phi(\theta) \\ \mathcal{S}_N(x) := \left|\frac{\sin(Nx/2)}{N\sin(x/2)}\right|\end{gathered}}$$

For quasiperiodic systems $\mathcal{F}_{\omega,N}$ converges to the projection onto the eigenspace corresponding to eigenfrequency ω, as $N\to\infty$.
For the L63 system,

$$\sup_{\omega\in\mathbb{R}\setminus\{0\}} \left\|\mathcal{F}_{\omega,N}\phi\right\|_{L^2(\mu)}^2 = O\left(N^{-2}\right) \quad \text{as } N\to\infty.$$

Outline III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The generator of the Koopman operator

$$U^t : L^2(\mu) \to L^2(\mu), \quad \left(U^t f\right) : x \mapsto f\left(\Phi^t x\right).$$

Vector field V as an operator

- $$(Vf)(x) := V(x) \cdot \nabla f(x)$$
- $$(Vf)(x) = \lim_{t\to 0} t^{-1}\left[f(\Phi^t x) - f(x)\right]$$
- $$V(f) := \lim_{t\to 0} t^{-1}\left[U^t f - f\right]$$

$$\boxed{U^t = e^{tV} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n V^n}$$

One-parameter subgroup

$$U^t : L^2(\mu) \to L^2(\mu), \quad (U^t f) : x \mapsto f\left(\Phi^t x\right).$$

U^t is a 1-parameter unitary group.

$$\boxed{U^t = e^{tV} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n V^n}$$

$$V(f) := \lim_{t \to 0} t^{-1}\left[U^t f - f\right], \quad C^1 \subset \mathrm{dom}(v) \subset_{\mathrm{dense}} L^2(\mu)$$

The spectral measure

$$\boxed{U^t : L^2(\mu) \to L^2(\mu), \quad Vf := \lim_{t\to 0} t^{-1}\left[U^t f - f\right], \quad U^t = e^{tV}.}$$

Spectral measure

An operator-valued measure

$$E : \mathrm{Borel}(\mathbb{R}) \to \text{Projection operators on} L^2(\mu)$$

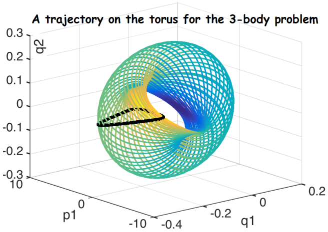
which assigns to every measurable set $U \subseteq \mathbb{R}$ a projection operator $E(U)$; $E(\mathbb{R}) = \mathrm{Id}$; $E(\varnothing) = 0$; and

$$E(U \cap V) = E(U)E(V) = E(V)E(U)$$

$$\boxed{V = \int_{\mathbb{R}} \iota\omega dE(\omega), \quad U^t = \int_{\mathbb{R}} e^{\iota t\omega} dE(\omega) = \int_{S^1} z^t d\tilde{E}(z).}$$

Spectral measure for quasiperiodic systems

For quasiperiodic dynamics, $\mathcal{D} = L^2(\mu)$, $\mathcal{D}^\perp = \{0\}$.

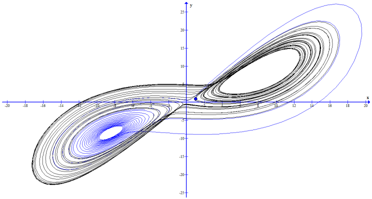


The spectral measure E is a discrete measure, composed of projections along each individual Koopman eigenfunction.

$$E(U) = \sum_{\omega_j \in U} \langle z_j, \cdot \rangle_{L^2(\mu)} z_j$$

Spectral measure for weakly mixing systems

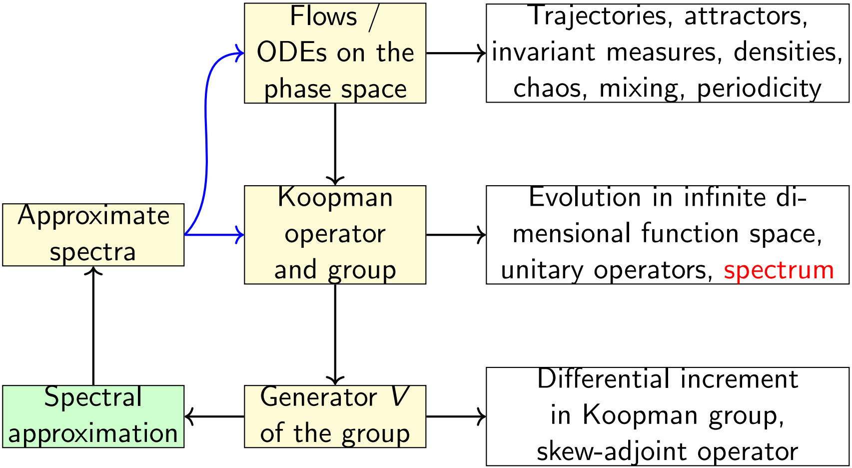
For weakly mixing / chaotic systems, $\mathcal{D} = \{\text{constants}\}$, $\mathcal{D}^{\perp} = L^2_0(\mu)$.



Spectral decomposition theorem

For systems such as the L63 attractor, the unitary group U^t is isomorphic to a multiplication operator on $L^2(S^1)$. On every cyclic space, U^t has a spectral density.

Outline IV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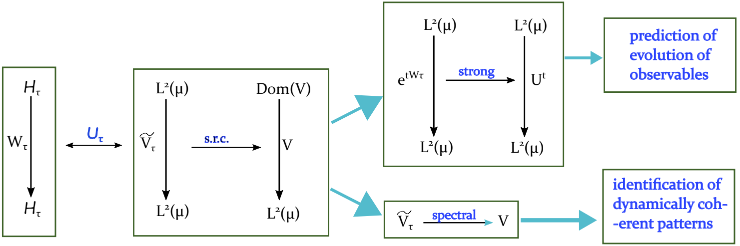
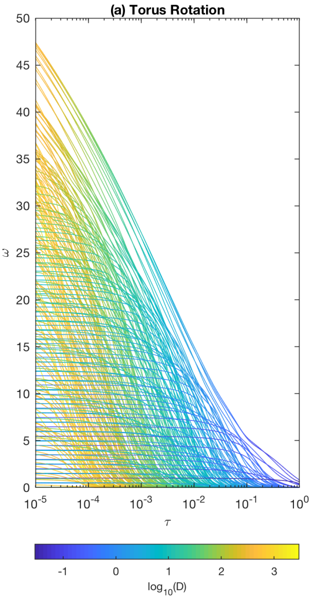
Spectral approximation scheme

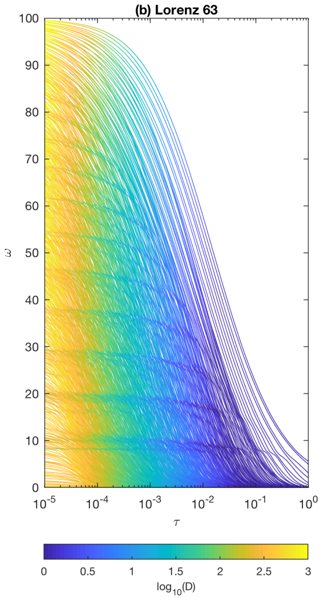


1. There is a family $\{\tilde{V}_\tau : \tau > 0\}$ of compact operators on $L^2(\mu)$ which converges to V on C^2 functions
2. The $\tilde{V}_\tau$ are conjugate to a compact, skew-adjoint operator W_τ on an associated Hilbert space $\mathcal{H}_\tau$.
3. The W_τ have complete eigenbasis of functions $\{\zeta_{\tau,j} : j\}$ and corresponding eigenvalues $\{\iota\omega_{\tau,j} : j\}$.

Reproducing kernel Hilbert space compactification of unitary evolution groups
- Giannakis, Das, Slawinska

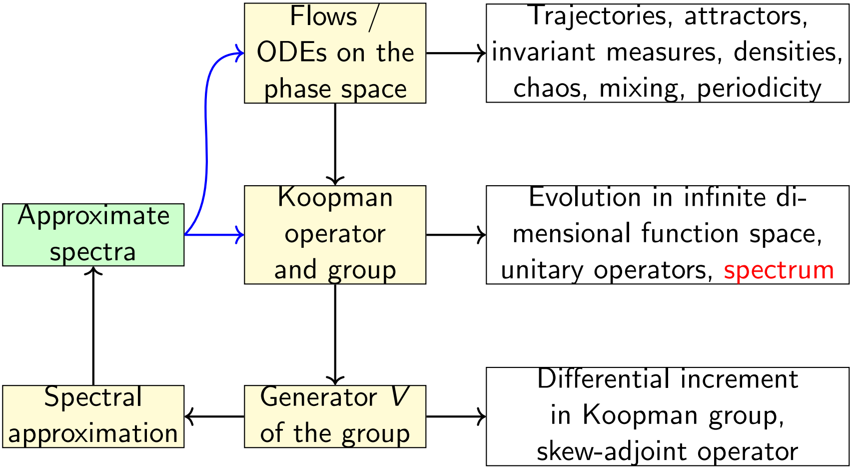


Theorem A. For every element $i\omega$, $\omega \in \mathbb{R}$, of the spectrum of the generator V, there exists a continuous curve $\tau \mapsto \omega_\tau$ such that $i\omega_\tau$ is an eigenvalue of W_τ, and $\lim_{\tau \to 0^+} \omega_\tau = \omega$.



Theorem B. For every interval $U = [a, b] \subset \mathbb{R}$ such that a, b are not eigenvalues, $E_\tau(U)$ converges strongly to $E(U)$ as $\tau \to 0^+$, in the strong operator topology.

Outline IV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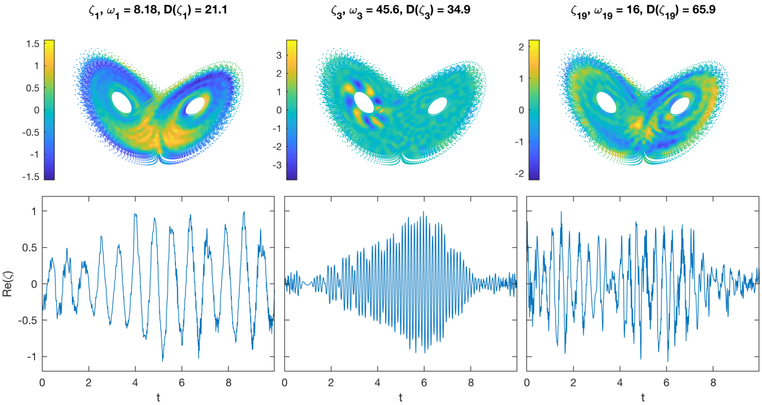
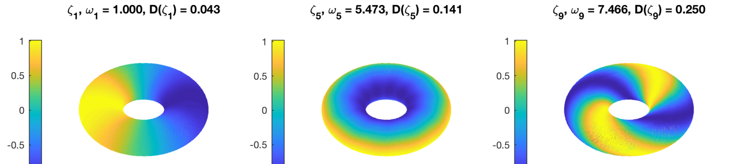
Coherent spatiotemporal patterns





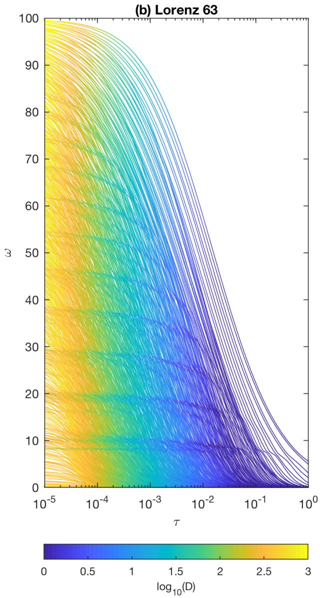
Approximate eigenfunctions

$$(U^t z)(x) = z(\Phi^t x) \approx e^{\iota\omega t} z(x)$$

More precisely, ω is an ϵ-approximate eigenfrequency upto time $T > 0$ if

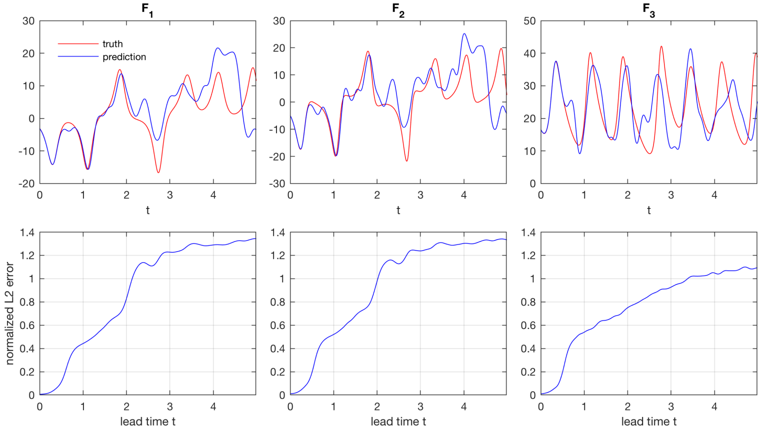
$$\left\| U^t z - e^{\iota\omega t} z \right\|_{L^2(\mu)} < \epsilon, \quad 0 \leq t \leq T$$

- True eigenfrequencies are approximate eigenfrequencies.
- Any $\omega \in \mathrm{supp}(E)$ is an approximate eigenfrequency, and
- for every ϵ, T, it has a corresponding approximate eigenfunction.



Theorem C. If the energy along a continuous curve $\tau \mapsto \omega_\tau$ remains bounded as $\tau \to 0^+$, the the associated ζ_τ converges to an approximate Koopman eigenfunction.

Predictions for the L63 attractor.



Theorem D : For fixed t, U^{tW_τ} converges strongly to $U^t = e^{tV}$, as $\tau \to 0^+$.

Summary

- The spectral measure is the underlying mechanism for many results in ergodic averaging and signal processing.
- The evolution of a measurement / observation under the flow is directly expressed by the Koopman group U^t. U^t in turn is determined by the spectral measure E.
- For chaotic systems, there are uncountable number of approximate eigenfunctions for arbitrary error ϵ and arbitrary lead-time t.
- Thus spectral theory supports the fact that chaotic systems show almost-periodic behavior at every time-scale.
- It is possible to approximate spectral measure by compactifying the generator.
- We used kernel-integral operators and associated RKHS for the compactification.

References

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Thank you