Local-global principles over semi-global fields

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Rational points and local-global principles (LGP)

Question: Given an F-variety V, is $V(F) \neq \varnothing$?

Related Question: Does V satisfy a local-global principle?
I.e.: $V(F) \neq \varnothing \Leftrightarrow (\forall v) V(F_v) \neq \varnothing$ (v: absolute values on F).

Classical situation: F a global field (e.g. $\mathbb{Q}$, $\mathbb{F}_p(x)$).

Ex: $Q \subset \mathbb{P}^n$ a quadric hypersurface ($q = 0$), q a quadratic form.
Q is a homogenous space for $\mathrm{O}(q)$, and $\mathrm{SO}(q)$ if $\dim(q) > 2$.
LGP holds by Hasse-Minkowski: q isotropic over F iff over all F_v.

Ex: V a torsor (principal homogeneous space) for PGL_n. These are classified by $H^1(F, \mathrm{PGL}_n)$; also classifies central simple F-algebras.
So LGP holds by Thm. of Albert-Brauer-Hasse-Noether
(a csa splits over F iff split over all F_v).

$\mathrm{SO}(q)$ and PGL_n are rational connected linear algebraic groups. In general, G-torsors satisfy LGP over a global field F if G is rational and connected (Sansuc–Chernousov).

Obstruction to LGP for torsors

Question: Given an algebraic group G over F, do all G-torsors satisfy LGP?

Obstruction: $Ш(F, G) = \ker\left[\alpha : H^1(F, G) \to \prod_v H^1(F_v, G)\right]$, the *Tate-Shafarevich set* (a group if G is commutative).

For F global: $Ш(F, G)$ is trivial if G rational and connected (as above) or G a finite gp. (Tchebotarev); finite if G linear alg. gp. (Borel–Serre; Conrad) and conjecturally if G an elliptic curve (Birch–Swinnerton-Dyer Conjecture).

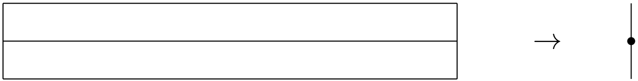
In global fn. field case, $F = k(C)$ for k finite, C sm. proj. curve; abs. val. $v \leftrightarrow$ discrete valuations on $F \leftrightarrow$ closed points of C.

Semi-global fields

A *semi-global field* is a one-variable function field F over a complete discretely valued field K; i.e., the function field of a curve over K.

Examples: $F = \mathbb{Q}_p(x)$, $F = k((t))(x)$, any finite extension of these.

Let $T = \mathcal{O}_K$ (e.g. $\mathbb{Z}_p$ or $k[[t]]$). Then there is a *regular model* $\mathcal{X}$ of F over T; i.e., a flat projective regular T-curve $\mathcal{X} \to \mathrm{Spec}(T)$ with function field F.



$F = K(x), \quad \mathcal{X} = \mathbb{P}^1_T$ $\qquad\qquad$ $\mathrm{Spec}(T)$

Local in one direction, global in the other: "semi-global".

LGP over semi-global fields

Several possible LGP's and obstructions to consider here:

$\text{Ш}(F,G) = \ker\left[H^1(F,G) \to \prod_v H^1(F_v,G)\right]$; v ranging over discrete val's on F corresp. to codim 1 points on models of F.

$\text{Ш}_X(F,G) = \ker\left[H^1(F,G) \to \prod_P H^1(F_P,G)\right]$; X the closed fiber of a model $\mathcal{X}$ of F; P the points of X; $F_P = \mathrm{frac}\,\widehat{\mathcal{O}}_{\mathcal{X},P}$.

Are these finite? trivial? related? Implications for alg. structures?

HHK (via patching methods): • $\text{Ш}_X(F,G) \subseteq \text{Ш}(F,G)$.
• G a *rational* connected lin. alg. group $/F \Rightarrow \text{Ш}_X(F,G)$ is trivial.
• $\text{Ш}_X(F,G)$ trivial $\Rightarrow$ LGP holds $\forall$ G-homogeneous spaces $/F$.

Consequences: • ABHN holds for csa's over semi-global fields:
A is split over $F \Leftrightarrow A$ is split over each F_P. (CPS: or each F_v)
• HM holds for quadratic forms of dim > 2 over semi-global fields:
q isotropic over $F \Leftrightarrow q$ is isotropic over each F_P. (CPS: or each F_v)

LGP for more general groups over semi-global fields

1) Say G not connected, but rational (each component rational$/F$).
Then $Ш_X(F, G)$ depends on the reduction graph Γ of $\mathcal{X}$:
configuration of components of the closed fiber:

HHK: $Ш_X(F, G) = \mathrm{Hom}(\pi_1(\Gamma), G/G^0)/\sim$, hence finite.
($\sim$ = conjugation) If G disconnected, get:
LGP holds $\Leftrightarrow \Gamma$ is a tree (no loops).

Ex.: HM for *binary* quadratic forms $/F \Leftrightarrow \Gamma$ is a tree.
(Note: Quadric hypersurface $Q : (q = 0)$ isn't a homogeneous space under $\mathrm{SO}(q)$; and $\mathrm{O}(q)$ is not connected.)

2) Say G is connected, but not rational. Then LGP does *not* necessarily hold: Example of CPS with a model $\mathcal{X}$ with generic fiber an elliptic curve; closed fiber consisting of 3 $\mathbb{P}^1$'s meeting cyclically (forming a triangle); G a non-rational torus over F.

Non-rational groups via the Rost invariant

Does LGP *ever* hold for non-rational connected groups?

Yes: HHK obtained examples of non-rational groups where LGP holds, via higher Galois cohomology:

For a field F, a linear algebraic group G over F, and an integer $m > 1$ that is prime to $\mathrm{char}(F)$, there is a canonical map

$$\rho : H^1(F, G) \to H^3(F, \mathbb{Z}/m\mathbb{Z}\,(2))$$

called the *Rost invariant*. ($\mathbb{Z}/m\mathbb{Z}\,(2) = \mu_m^{\otimes 2}$)
Often ρ is known to have trivial kernel (e.g., a split almost simple group of type B_6, D_6, D_7, E_6, E_7.)

For a semi-global field F, HHK proved LGP for $H^3(F, \mathbb{Z}/m\mathbb{Z}\,(2))$.

Conclusion: For F semi-global, and groups G with $\ker(\rho)$ trivial (even if G is not rational), LGP holds for G-torsors over F.

LGP for other non-rational connected groups

Conjecture (CPS): $\text{Ш}(F, G)$ is trivial if G is a semisimple simply connected group over the function field F of a p-adic curve (F semi-global with $K/\mathbb{Q}_p$ finite).

Open in general; known in many cases: Hu, Preeti, Parimala, Suresh. What if K is a more general cdvf? Must LGP hold if G is semisimple simply connected?

Not necessarily: Examples of HHK+CPS of a cdvf K, a semi-global field F/K, with $\text{Ш}(F, G)$ non-trivial, where G is a semisimple simply connected group (and can even take G to be defined over $T = \mathcal{O}_K$). By taking the residue field of T large enough (infinite transcendence degree over $\mathbb{Q}$), can have $\text{Ш}(F, G)$ *infinite*.

Can do the same for tori G defined over T (vs. earlier example of CPS, defined over F).

Positive results for groups defined globally

Suppose a linear algebraic group G is defined *globally*: not just over F, but over $\mathcal{X}$ or even over $T = \mathcal{O}_K$. HHK+CPS:

1) Say G is connected and reductive over $\mathcal{X}$ (e.g., a torus, or semisimple, like $\mathrm{SL}_1(D)$). Then $\text{Ш}(F, G) = \text{Ш}_X(F, G)$.

2) Say G connected and reductive over $T = \mathcal{O}_K$, and let $k = T/\mathfrak{m}$. Suppose X is reduced. Suppose Γ is a "geometric tree"; i.e. a tree after any base change k'/k (plus a bit more if $\mathrm{char}(k) = p$: $(*)_\Gamma$, $(*)_G$). Then $\text{Ш}(F, G) = \text{Ш}_X(F, G)$ is trivial.

Can also drop the reductive hypothesis if $K = k((t))$ with $\mathrm{char}(k) = 0$ and G is defined over k.

If we drop the connectivity hypothesis also, then under these hypotheses we still get $\text{Ш}_X(F, G)$ is trivial.

Exact determination of $\text{Ш}(F, G)$

For G a group$/k$, $P, Q \in G(k)$ are R-*equivalent* if $\exists\ U \subset \mathbb{P}^1_k$ containing P, Q. The R-equivalence class of $\mathrm{id} \in G(k)$ is a normal subgroup R of $G(k)$, and $G(k)/\mathrm{R}$ = group of R-equivalence classes. (Related to rational connectedness if k is alg. closed.)

Let F be semi-global over a cdvr T; $\mathcal{X}$ a regular model; $k = T/\mathfrak{m}$.

HHK+CPS: Say G connected and reductive over T (and $(*)_G$ if $\mathrm{char}(k) = p$). Suppose X consists of copies of $\mathbb{P}^1_k$ intersecting normally at k-points, st Γ has m cycles ($H^1(\Gamma, \mathbb{Z}) = \mathbb{Z}^m$). Then $\text{Ш}(F, G) = \mathrm{Hom}(\pi_1(\Gamma), G(k)/\mathrm{R})/\!\sim\; = (G(k)/\mathrm{R})^m/\!\sim$.

So LGP holds iff the graph is a tree or G is R-connected (just one R-equivalence class).

In general for G reductive and defined over T: still open.