Solar Panels, Energy and Area Under the Curve

Victor J. Donnay, Bryn Mawr College

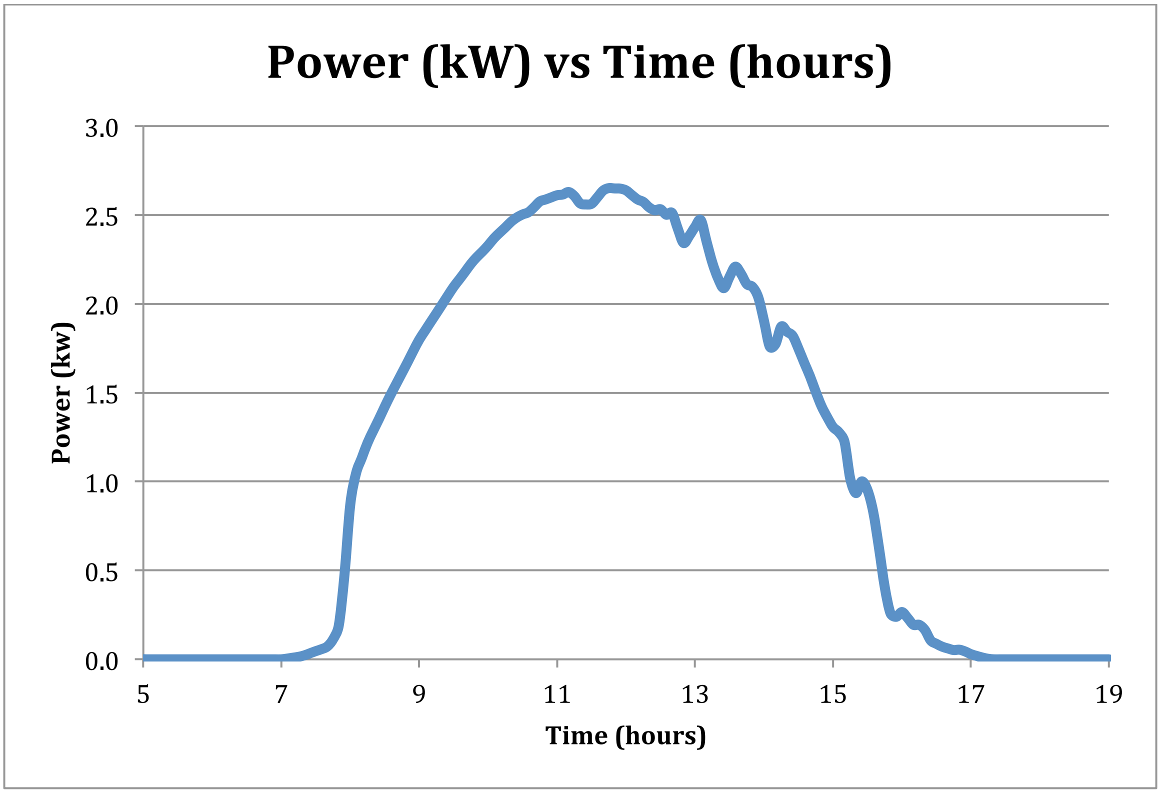

Figure 1. The power (in kW) produced by a solar panel installation at Bryn Mawr College [1] on January 27, 2013.

Lesson Overview: Determine the total energy produced by a solar panel array over the course of a day by finding the area under the power vs time graph (see Figure 1).

Level: This unit is appropriate for students in a pre-calculus or calculus class. The material up until problem (9c) could also be accessible to lower level students.

Common Core State Standards for Mathematics [2]: Standard for Mathematical Practice 4 - Model with mathematics.

AP Calculus Goals [3]: Students should

- Understand the meaning of the definite integral both as a limit of Riemann sums and as the net accumulation of change, and should be able to use integrals to solve a variety of problems.
- Understand the relationship between the derivative and the definite integral as expressed in both parts of the Fundamental Theorem of Calculus.
- Be able to communicate mathematics and explain solutions to problems both verbally and in written sentences.
- Be able to model a written description of a physical situation with a function, a differential equation, or an integral.

- Be able to use technology to help solve problems, experiment, interpret results, and support conclusions.

Associated Materials: The spreadsheet Power Data contains the data used to create the graph as well as the calculations needed to determine the energy generated.

Mathematics Content:

1. Graphs provide information about the real world; in this case they show the amount of power (measures in watts W and kilowatts kW) that a solar panel array generates. Graphs can be generated by data that is given in a table.
2. When a graph represents a physical quantity, the area under the graph can have an important contextual meaning; in the case of the graph of power vs time, the area under the graph gives the energy generated by the solar panels.
3. Units are important in real world problems and keeping track of units can help one better understand math concepts.
4. The area under a curve can be approximated by adding up the areas of geometrical shapes.

Sustainability Content:

Alternative energy sources [4, 5], such as solar and wind power provide renewable energy that does not generate greenhouse gas (CO_2) emissions and are becoming more common around the country. Solar power can be harnessed in a variety of ways including solar hot water heating, photovoltaic cells (solar panels) and concentrated solar [6, Ch 6 and Ch. 25]. In this unit we focus on energy generated by solar panels. Different areas in the country receive different amounts of sunshine; maps that show this data are available at [7]. These panels often include measuring devices which record the amount of power that is being generated every few minutes. This data can be displayed and viewed in real time [1] to see how much renewable energy is being produced.

Solar Panels, Energy and Area Under the Curve Lesson

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Goal: Determine the total energy produced by a solar panel array over the course of a day by using the graph of power vs time (see Figure 4).

Figure 1. Solar Panels.

1. What do you know about solar power?
2. Looking at the accompanying utilities bill for electric residential service, how much electricity was used during the current month? What units are used to describe the amount of electrical energy? How much was the electrical bill for the month?
3. What are the units for power and for energy and how are they related?
4. a. If a household is using 3 kW (kilowatt) of power continuously from 1pm to 5 pm (see Figure 2), how much energy is used?
   b. What is the area = height x width under the power curve for $1 \leq t \leq 5$? Give the units for this area that you get by multiplying the units for the height by the units for the width.

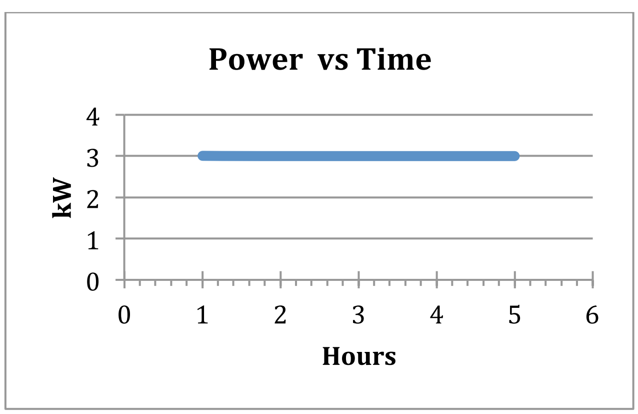

Figure 2. Energy usage with constant power.

5. a. If the household uses 2 kW of power from 1pm to 3pm, then 4 kW from 3pm to 7pm and 1 kW from 7 pm to 9 pm (see Figure 3), how much energy does it use?
   b. What is the area = height x width under the power curve for 1 $\leq$ t $\leq$ 9? Give the units for this area that you get by multiplying the units for height by the units for the width.

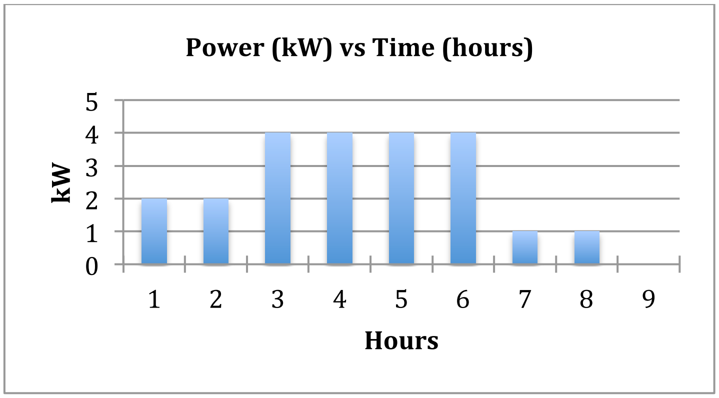

Figure 3a. Energy usage with piecewise constant power.

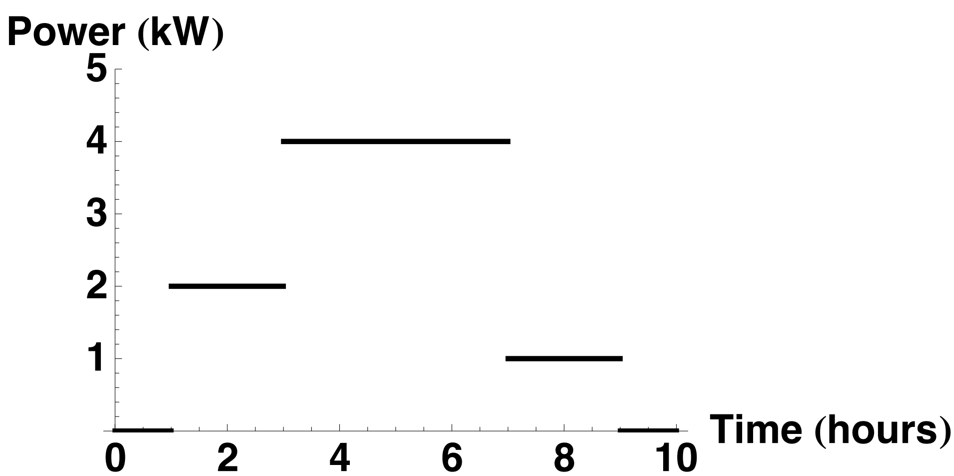

Figure 3b. Energy usage with piecewise constant power.

In Figure 4, we plot the power (kW) produced by a solar panel installation at Bryn Mawr College [1] on January 27, 2013 as a function of time. The value of the power was recorded every 5 minutes so we have a plot of a discrete set of points rather than of a continous curve. We have plotted the data starting at 5 am and continuing until 7 pm (19 hundred hours).

6. At 9 am, how much power is being produced by the solar panels?
7. Estimate the time of sunrise and sunset on January 27, 2013 in Bryn Mawr, Pa.
8. The graph consists of values plotted at 5 minute intervals. On Figure 4, draw a continuous curve that fits the data.

9. Examine the power vs time graph generated by the solar panels (Figure 4).
   a. What is the maximum power that the solar panels generated during the day? At what time of day did that maximum occur?
   b. If the panels had been able to produce that maximum amount of power from sunup to sunset, how much energy would they have produced?
   c. Estimate how much energy the solar panels actually produced.
   d. Why are there some dips and wiggles in the graph?

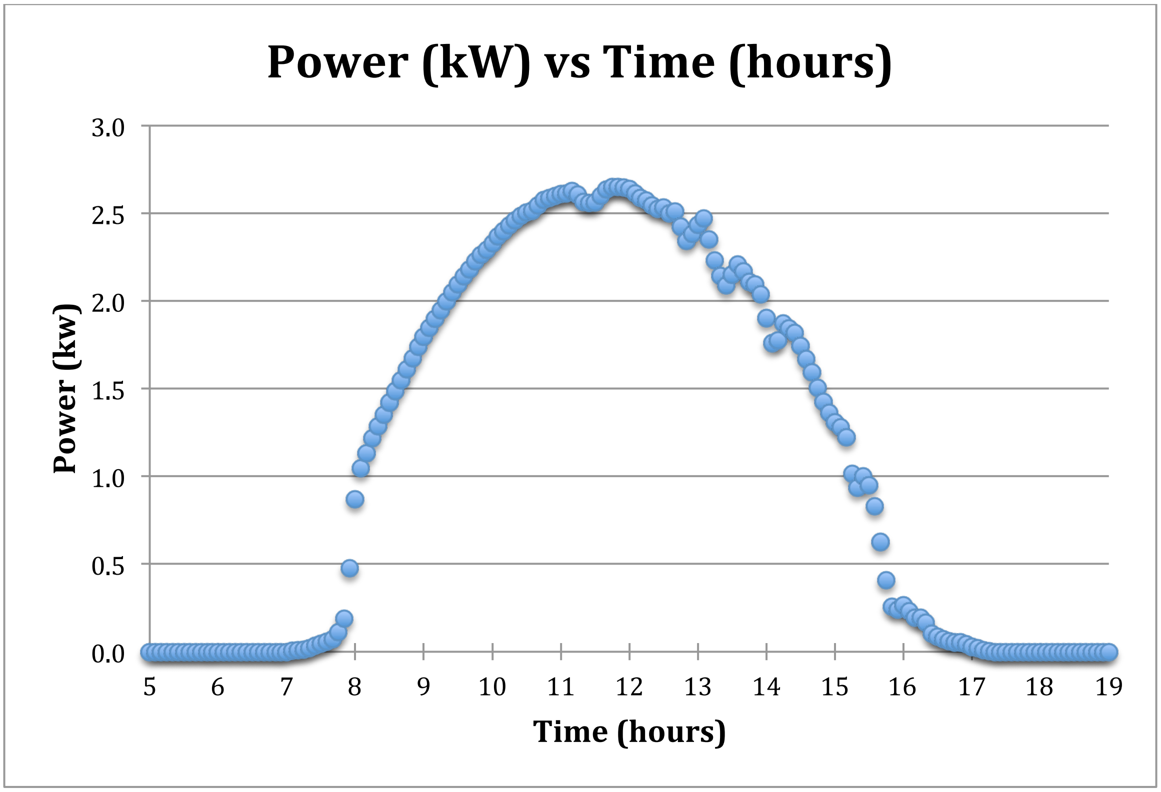

Figure 4. The power (kW) produced by a solar panel installation at Bryn Mawr College on January 27, 2013. http://sustainability.blogs.brynmawr.edu/2012/11/13/first-solar-panels/.

10. Do you think it would be “worth it” home to install solar panels at your school (or home)? What information would you need to be able to answer this question? Present your findings to the school board (or college administration).

Lesson Plan Notes

0. Lesson Launch. Start the unit by engaging the students in a discussion about solar energy. Ask the students what they know about solar energy.

   Is there a solar array in your town? Take the students for a visit. Or bring a solar installation professional or other knowledgeable person to visit the school and talk about solar energy and demonstrate a solar panel. Maybe your schools science teacher has a small solar panel or solar cell. Have the students look at pictures or videos about solar energy.

1. Electrical Energy Background. The lesson is based on using the units of power (watts = W) and energy (kilowatt hours = kWh). These units can be very confusing (for all of us)!

   Introduce students to these units by having them read an energy bill. If possible, have them bring an energy bill from home. The lesson kit includes a bill that one can use. In the bill, one finds that the units of electrical energy are **kWh** which stands for kilowatt hours. In the sample bill, the household is being billed for using 1179 kWh which costs $202.

2. A good place to start discussing energy and power is with a 100-watt light bulb which you can show to the students. If ten of these bulbs are on at the same time, they are using 10 x 100 watts = 1000 watts = 1 kW of power.

   Power is the rate at which energy is flowing through the system. To help remember that power is a rate of change, it is useful to recall the definition of a watt:

   $$1 \text{ watt} = 1 \text{ joule per second } = \frac{1\text{ joule}}{\text{sec}}.$$

   Without getting into the precise definition of joule, suffice it to say that joule is a unit of energy and hence we see that watt is a rate of change of energy. In the context of calculus, power is the derivative of energy and energy is the integral of power.

   The relation of power and energy is that

   $$\text{energy} = \text{power x time}.$$

   Thus if we turn on our ten one-hundred watt bulbs and keep them on for 3 hours, the amount of energy they use is

   $$1 \text{ kW x 3 hours} = 3 \text{ kilowatt hours} = 3 \text{ kWh}.$$

   Practice Example: If we have 3 one-hundred watt light bulbs and they run for 4 hours, how much energy do they use?

   $$300 \text{ W x 4 hours} = 1200 \text{ W hours} = 1.2 \text{ kW hours} = 1.2 \text{ kWh}.$$

   We could also do the calculation by converting the 300 W into kW

   $$300 \text{ W} \times \frac{1 \text{ kW}}{1000 \text{ W}} = .3 \text{ kW}.$$

   Then the energy used is

$$.3 \text{ kW} \times 4 \text{ hours} = 1.2 \text{ kWh}.$$

A nice introduction to sustainable energy issues is the book *Sustainable Energy without the Hot Air,* by David MacKay that is available free online at http://www.withouthotair.com/. He gives a short primer about power and energy [6, p. 24] in which he compares power to the flow rate of water and energy to the total amount of water.

Consider partnering with one of your science teachers for this lesson. She might have good ideas for how to help the students understand about power (W) and energy (kWh).

4. Area Under the Curve.

   For a problem with constant power, if we plot the power (kW) on the vertical axis and the time (hours) on the horizontal axis, then the

   energy = power x time

   can be interpreted as the **area under the curve** :

   Energy = .3 kW x 4 hours = 1.2 kWh
   = height (kW) x width (hrs)
   = area under the power curve in units of kWh.

   Note that the units for energy (kWh) come about by multiplying the unit of height (kW) by the unit of width (hours). So one can interpret the energy as the area under the power curve.

5. If the power is piecewise constant, the energy can still be interpreted as the area under the curve. Now one must split the area calculation into three parts.

6. From examining the graph, the students should see that at 9 am, the solar panels are generating about 2 kW of power. It is important that the students include units with their answer.

7. The solar panels start generating power just after sunrise (about 7:30 am) and continue until sunset (about 17 = 5pm).

8. The power graph is a discrete graph. It only has values at 5 minute intervals. One can fit a continuous curve to these data points, resulting in a figure like that at the start of the lesson packet. In math classes, students usually work with graphs that are given by continuous functions defined at formula and defined for all time values. In this problem, there is no such formula for the function. The function is defined by the values of the data.

9. a. The maximum power during the day occurred at about noon and was about 2.6 kW.

   b. If the solar panel had been able to generate 2.6 kW of power for the 9.5 hours of the day, it would have produced

$$2.6 \text{ kW} \times 9.5 \text{ hours} = 24.7 \text{ kWh}.$$

c. **Area Under the Curve**: Even when the power curve is not constant, as in Figure 4, the energy produced can still be interpreted as the area under the curve. One can think of approximating the area under the curve by lots of very narrow rectangles. We now have a piecewise constant function so its area is the energy generated by these constant values of power. As the number of rectangles becomes larger (i.e. goes to infinity) and their width goes to zero, the area under the rectangles approaches the area under the curve. This limiting process is the base for the definition of the integral as the limit of Riemann sums. There are interactive applets [8] on the web that let the user experiment with Riemann sums with various numbers of boxes.

To find the energy generated by the solar panels, students can either approximate the area using geometric shapes such as rectangles and triangles or can use the accompanying spreadsheet (Power Data) with the actual data to calculate more precisely the area by summing up the area of a large number of rectangles.

The page Energy Calculation in the spreadsheet shows how to calculate the sum of the areas of the boxes. The height of each box is given by the power. The width of the box is the time interval between measurements = 5 min = 1/12 of an hour.

The energy produced by the solar panels of the course of the day was 16.0 kWh.

d. This power graph was generated on a beautiful sunny winter day. The few wiggles in the graph are due to an occasional cloud. If you go to the Bryn Mawr College solar panel website [1] and scroll down to the Graphs section, you can see the power graphs for the past 24 hours and by moving the dark rectangle slider, you can zoom in any of the power graphs for the past week and also see how much energy was generated in each 24 hour period. You will see that it is rare to get such a smoothly shaped graph!

10. To determine whether installing solar panels are "worth it" might require knowing: the cost to buy and install the solar panels, the cost for upkeep (minimal) and the price of electricity in the region. Are there any tax rebates or other financial incentives for purchasing solar panels? With this information, one could calculate the pay back time. One might also take into account the reduced carbon footprint that would arise from using solar panels. This reduction in greenhouse gas output could be of benefit to the environment and society although at present there is not a financial savings associated with it. If society were to institute some form of carbon tax, then there would be additional finance benefits to the solar panels.

Calculus Connections: Integration

In calculus, students learn to calculate the area under a curve given by a function y = f(t) by taking the integral $\int_a^b f(t)dt$. They usually evaluate the integral by taking the anti-derivative of f. In problem 9d, they may want to follow this approach. Unfortunately, the graph is not given by a function. It is made by plotting discrete data points and then joining up the points to make a curve. **There is no function given by a formula** and thus no anti-derivative for the students to use. However they can still approximate the value of the integral by estimating the area under the curve in a variety of ways, including using Riemann sums.

Spreadsheet Usage.

The power graph (Figure 4) was generated from the data in the Power Data spreadsheet. To recreate the graph (in Excel or in Googledocs for instance), highlight the two columns, then go to the **Chart menu**, choose **Scatter** and then choose one of the graphing options under Scatter.

One can use the data in the spreadsheet to calculate a Riemann sum that gives the area under the curve and hence the energy generated by the solar panels. The power generated by the solar panels was recorded at five minute intervals. If we assume that the power over that five minute interval is constant at a value P (kW), then the energy generated over those 5 minutes = $\frac{1}{12}$ hour is

$$\mathrm{P\ kw} \times \frac{1}{12}\ \mathrm{hour} = \mathrm{P} * \frac{1}{12}\ \mathrm{kwh}.$$

In the third column of the spreadsheet, we calculate the energy values. There is a quick and very cool way to do this that involves entering a formula for the energy in the first cell. Then clicking and dragging this first cell down the column will give the energy in all the other cells. Summing up the energy values over all the time intervals gives the total energy produced. This method is illustrated on the spreadsheet page Energy Calculation.

Extension activities:

1. Have the students visit the solar panel website [1] and describe what the weather would have been like on each of the days and how much energy was generated. They will find that there is a big difference in energy generated between a sunny and cloudy day.
2. Partner with a social studies or government teach to discuss government policy towards alternative energy in your state.
3. Partner with a science teacher for a hands-on lesson about power. Use a watt meter to determine how much power various appliances in the school use.
4. If your school or an organization in the community has solar panels with data output, use your local data for this lesson.

References:

1. Information about the solar panel installation at Bryn Mawr College, http://sustainability.blogs.brynmawr.edu/2012/11/13/first-solar-panels/. Graphs of the daily power product can be seen at https://enlighten.enphaseenergy.com/public/systems/faZD132956
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