Relative Reshetikhin-Turaev invariants, hyperbolic cone metrics and discrete Fourier transforms

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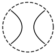
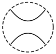
Given an odd integer $r = 2N + 1 \geq 3$. Consider the primitive r-th root of unity $q = e^{\frac{2\pi i}{r}}$.

- The Kauffman bracket for a framed link L with a plane diagram D, denoted by $\langle D \rangle$, is defined by the following rules.
  - Kauffman bracket skein relation:

⟨  ⟩ $= e^{\frac{\pi\sqrt{-1}}{r}}$ ⟨  ⟩ $+ e^{-\frac{\pi\sqrt{-1}}{r}}$ ⟨  ⟩

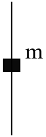
  - Framing relation:

$$\langle L \coprod O \rangle = \left(-e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}}\right)\langle L \rangle$$

- Special diagrams (linear combination of diagrams):

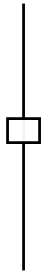
m-th Jones-Wenzl idempotent e_m

(black box)



Kirby coloring Ω_r

(white box)



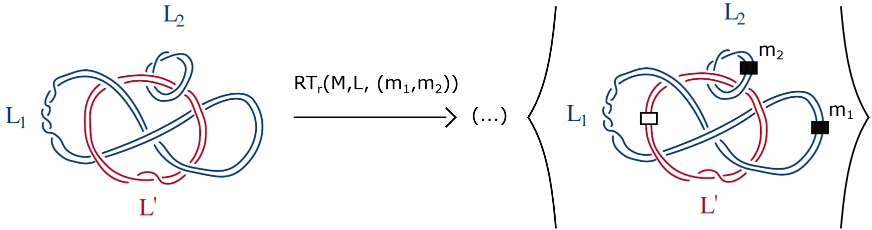
$SO(3)$ relative Reshetikhin-Turaev invariants for the pair (M, L)

Let $I_r = \{0, 2, 4, \ldots, r-3\}$ be the set of even integers in between 0 and $r-2$.

For a closed, oriented 3-manifold $M = \mathbb{S}^3_{L'}$ obtained by doing surgery along the framed link $L' \subset \mathbb{S}^3$, $L = L_1 \cup \ldots L_n \subset M$ framed link, $\mathbf{m} \in I_r^n = \{0, 2, \ldots, r-3\}^n$,

- the $\mathbf{m}$-th relative Reshetikhin-Turaev invariants (in short, relative RT invariants) of the pair (M, L) is defined by

$$RT_r(M, L, \mathbf{m}) = (\ldots)\langle e_{m_1}, \ldots, e_{m_n}, \Omega_r, \ldots, \Omega_r \rangle_{D(L \cup L')}$$



$SO(3)$ relative Reshetikhin-Turaev invariants for the pair (M, L)

In particular,

- When $M = \mathbb{S}^3$, then

$$RT_r(\mathbb{S}^3, L, \mathbf{m}) = (\dots)\langle e_{m_1}, \dots, e_{m_n}\rangle_{D(L)} = (\dots)J_{\mathbf{m}}(L, t = q^2 = e^{\frac{2\pi i}{N+\frac{1}{2}}})$$

- When $\mathbf{m} = \mathbf{0}$, then

$$RT_r(M, L, \mathbf{0}) = RT_r(M)$$

Q: What is the topological and geometrical meaning of the invariants?

Volume conjecture: a study about the asymptotics of quantum invariants and the relationship with topology/geometry

Quantum invariants	$RT_r(M, L, \mathbf{N})$ $\left(= J_{\mathbf{N}}(L, e^{\frac{2\pi i}{N+\frac{1}{2}}}), M = \mathbb{S}^3\right)$	$RT_r(M, L, \mathbf{m})$	$RT_r(M, L, \mathbf{0})$ $(= RT_r(M))$
Hyperbolic Geometry	?	?	?

Volume conjecture for the relative Reshetikhin-Turaev invariants

Conjecture: (W.-Yang '20)

For a sequence $\mathbf{m}^{(r)} = (m_1^{(r)}, \dots, m_n^{(r)}) \in I_r^n = \{0, 2, \dots, r-3\}^n$, let

$$\theta_k = \left| 2\pi - \lim_{r\to\infty} \frac{4\pi m_k^{(r)}}{r} \right| \in [0, 2\pi]$$

and let $\theta = (\theta_1, \dots, \theta_n)$. If M_{L_θ} is a hyperbolic cone manifold consisting of M and a hyperbolic cone metric on M with singular locus L and cone angles θ, then

$$\lim_{\substack{r\to\infty \\ r \text{ odd}}} \frac{4\pi}{r} \log \mathrm{RT}_r(M, L, \mathbf{m}^{(r)}) = \mathrm{Vol}(M_{L_\theta}) + \sqrt{-1}\mathrm{CS}(M_{L_\theta}) \qquad \text{mod } \sqrt{-1}\pi^2\mathbb{Z},$$

where $\mathrm{Vol}(M_{L_\theta})$ and $\mathrm{CS}(M_{L_\theta})$ are the hyperbolic volume and the Chern-Simons invariants of the cone manifold M_{L_θ}.

Relationship with Kashaev-Murakami-Murakami and Chen-Yang Volume conjectures

Recall that $r = 2N+1$, $\theta_k = \left| 2\pi - \lim_{r\to\infty} \frac{4\pi m_k^{(r)}}{r} \right| \in [0, 2\pi]$

- When $M = \mathbb{S}^3$ and $\mathbf{m} = \mathbf{N}$, since $RT_r(\mathbb{S}^3, L, \mathbf{N}) = (\dots) J_{\mathbf{N}}(L, e^{\frac{2\pi i}{N+\frac{1}{2}}})$ and $\theta = (0, \dots, 0)$, the conjecture suggests that for a hyperbolic link L, we have

$$\boxed{\lim_{N\to\infty} \frac{2\pi}{N+\frac{1}{2}} \log J_{\mathbf{N}}(L, e^{\frac{2\pi i}{N+\frac{1}{2}}}) = \mathrm{Vol}(\mathbb{S}^3 \backslash L) + \sqrt{-1}\,\mathrm{CS}(\mathbb{S}^3 \backslash L)}$$

- When $\mathbf{m} = \mathbf{0}$, since $RT_r(M, L, \mathbf{0}) = RT_r(M)$ and $\theta = (2\pi, \dots, 2\pi)$, the conjecture suggests that for every closed, oriented hyperbolic 3-manifold M with finite volume,

$$\boxed{\lim_{\substack{r\to\infty \\ r \text{ odd}}} \frac{4\pi}{r} \log\left(RT_r(M)\right) = \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)}$$

Volume conjecture for the relative Reshetikhin-Turaev invariants

Quantum invariants	$RT_r(M, L, \mathbf{N})$ $\left(= J_{\mathbf{N}}(L, e^{\frac{2\pi i}{N+\frac{1}{2}}}), M = \mathbb{S}^3\right)$	$RT_r(M, L, \mathbf{m})$	$RT_r(M, L, \mathbf{0})$ $(= RT_r(M))$
Hyperbolic Geometry	cusp manifold (complete str.) Kashaev-Murakami -Murakami VC	cone manifold (incomplete str.)	closed manifold (Dehn filling) Chen-Yang VC
Cone angle	0	$\theta = \left\lvert 2\pi - \lim_{r \to \infty} \frac{4\pi m^{(r)}}{r} \right\rvert$	2π

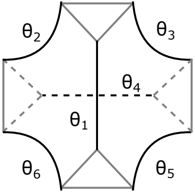
Fundamental shadow links

Fundamental shadow links: $\{L_{FSL} \subset M_c = \#^c \mathbb{S}^2 \times \mathbb{S}^1\}$ (FSL pair (M_c, L_{FSL}))

- [Costantino-Thurston] Any compact oriented 3-manifold with toroidal or empty boundary can be obtained from a suitable fundamental shadow link complement by doing an integral Dehn-filling to some of the boundary components.
- the $\mathbf{m}$-th relative RT invariants of $(M_c, L_{FSL}, \mathbf{m})$ is given by

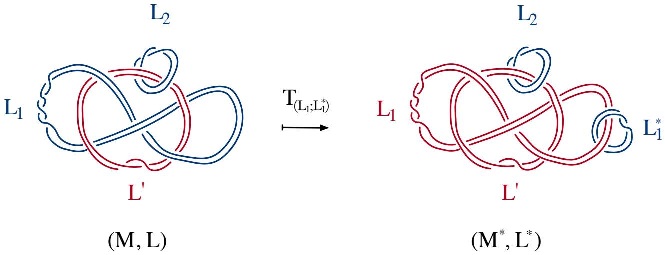
$$RT_r(M_c, L_{FSL}, \mathbf{m}) \sim \prod \text{Quantum 6j-symbols}$$

$$\underbrace{\begin{vmatrix} m_1^{(r)} & m_2^{(r)} & m_3^{(r)} \\ m_4^{(r)} & m_5^{(r)} & m_6^{(r)} \end{vmatrix}_{q=e^{\frac{2\pi\sqrt{-1}}{r}}}}_{\text{Quantum } 6j\text{-symbol}} \xrightarrow[\text{[Constantino]}]{\text{exp. growth rate}} \text{Vol of}$$



- [Belletti-Detcherry-Kalfagianni-Yang] The VC for $RT_r(M_c, L_{FSL}, \mathbf{N})$ is true.

Change-of-pair operation



Given (M, L), choose some components of the frame link L.

Step 1: do surgeries along those components;

Step 2: remove the curves isotopic to the core curves of the Dehn-filled tori in Step 1.

Proposition

If (M, L) and (M^*, L^*) are related by a change-of-pair operation, then $M\backslash L \simeq M^*\backslash L^*$.

Moreover, if (M, L) and (M^*, L^*) are two pairs so that $M\backslash L \simeq M^*\backslash L^*$, then they are related by a sequence of change-of-pair operations.

Main results

Theorem 1 (W.-Yang '20)

Conjecture is true for all pairs (M, L) obtained by doing a change-of-pair operation from the pair (M_c, L_{FSL}), for small cone angle.

Theorem 2 (Pandey-W. in progress)

Conjecture is true for all pairs (M, L) obtained by doing any sequence of change-of-pair operations from (M_c, L_{FSL}), for small cone angle.

Remarks:

- M in Theorem 1 & 2 cover all closed, oriented 3-manifold M.
- If we can push the cone angle from small all the way to 2π, then we can prove the Chen-Yang volume conjecture for the Reshetikhin-Turaev invariants for closed, oriented 3-manifold.

Main results

Quantum invariants	$RT_r(M, L, \mathbf{N})$ $\left(= J_{\mathbf{N}}(L, e^{\frac{2\pi i}{N+\frac{1}{2}}}), M = \mathbb{S}^3\right)$	$RT_r(M, L, \mathbf{m})$	$RT_r(M, L, \mathbf{0})$ $(= RT_r(M))$
Hyperbolic Geometry	cusp manifold (complete str.) Kashaev-Murakami -Murakami VC	cone manifold (incomplete str.)	closed manifold (Dehn filling) Chen-Yang VC
Cone angle	0	$\theta = \left\lvert 2\pi - \lim_{r\to\infty} \frac{4\pi m^{(r)}}{r} \right\rvert$	2π

Theorem 3 (W.-Yang '20)

Conjecture is true for all pairs obtained by doing any sequence of change-of-pair operations on $(\mathbb{S}^3, 4_1)$ for all cone angle $\theta \in [0, 2\pi]$, except finitely many cases.

Discrete Fourier Transform

Topology	Geometry	Quantum topology
Change-of-pair operations $(M, L) \mapsto (M^*, L^*)$	Different cone metrics $\downarrow$	Discrete Fourier transform $RT_r(M, L, \_) \mapsto RT_r(M^*, L^*, \_)$
Same complement	Same cusp metric	Same Turaev-Viro invariants

Poisson summation formula for the Discrete Fourier transform (W.-Yang '20)

$$\sum_{\mathbf{m}} |RT_r(M, L, \mathbf{m})|^2 = \sum_{\mathbf{n}} |RT_r(M^*, L^*, \mathbf{n})|^2.$$

Thank you!

Appendix: definition of the discrete Fourier transform

Let $f : I_r^n \to \mathbb{C}$, (I, J) be a partition of $\{1, \dots, n\}$ and let $\mathbf{n}_I = (n_i)_{i \in I}$ be a $|I|$-tuple of elements of I_r. Then the $\mathbf{n}_I$-th partial discrete Fourier coefficient of f is the function

$$\widehat{f}(\mathbf{n}_I) : \mathrm{I}_r^{|J|} \to \mathbb{C}$$

defined for all $\mathbf{m}_J$ in I_r^J by

$$\widehat{f}(\mathbf{n}_I)(\mathbf{m}_J) = \mu_r^{|I|} \sum_{\mathbf{m}_I} \left(\prod_{i \in I} \mathrm{H}(n_i, m_i) \right) f(\mathbf{m}_I, \mathbf{m}_J),$$

where the sum is over all $|I|$-tuples $\mathbf{m}_I = (m_i)_{i \in I}$ of elements of I_r, and

$$\mathrm{H}(n, m) = (-1)^{m+n} \frac{q^{(n+1)(m+1)} - q^{-(n+1)(m+1)}}{q - q^{-1}}.$$