Pedagogical Mathematical Practices as a way to Develop Pedagogy from Mathematics Coursework

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Joint Mathematics Meetings (Virtual)

SS64: Mathematics Courses Designed to Develop Mathematical Knowledge for Teaching High School

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Secondary Teacher Preparation

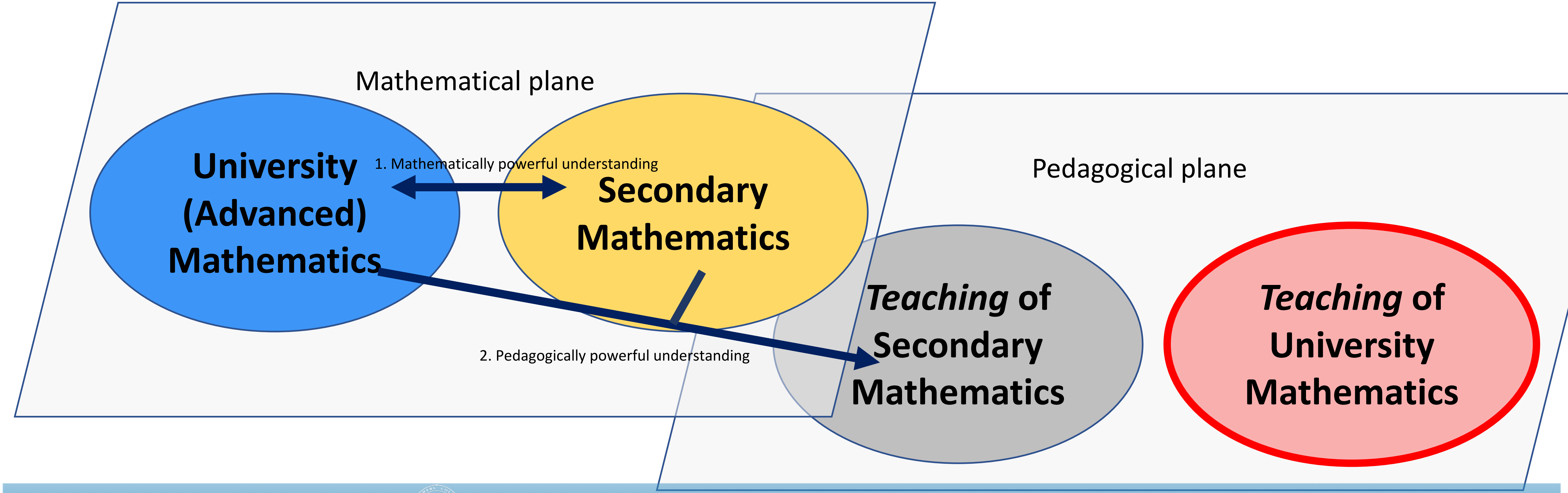
- **Tension** in secondary teacher preparation:
  - On one hand, secondary mathematics teachers need a sufficiently deep and robust knowledge of mathematics (e.g., content knowledge)
  - On the other hand, strictly mathematical ideas are not the only aspect for which secondary teachers need preparation (e.g., pedagogical knowledge, pedagogical content knowledge)

Secondary Teacher Preparation

- This is particularly challenging in (advanced) **mathematics coursework**
  - Driving question: *Are there ways to teach advanced mathematics courses that can be more effective mathematical preparation for secondary teachers?*
  - Specific challenge: *How can we meaningfully discuss pedagogy, and aim to develop teachers' pedagogical practice, in an advanced mathematics courses, when the aims are (and should be) about mathematics?*

Broad framing ideas

- Broadly, there are several big domains being considered:



ULTRA

Upgrading Learning for Teachers in Real Analysis

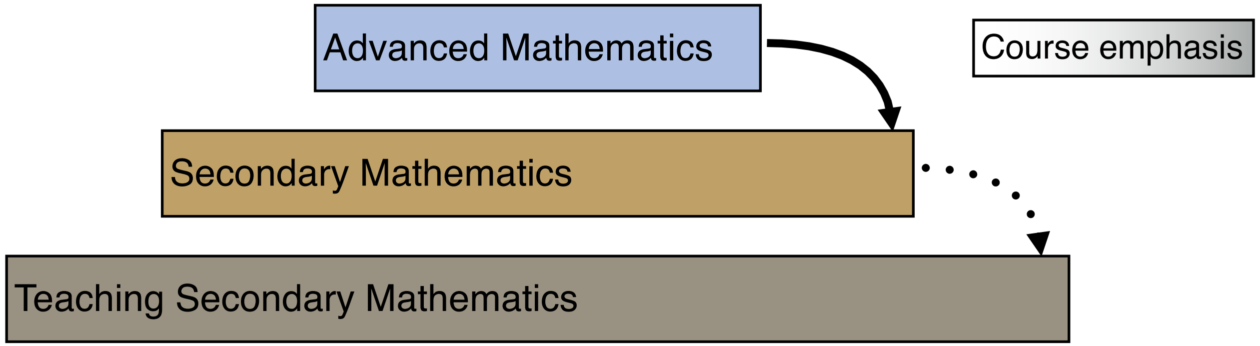
ULTRA

- A collaborative NSF-funded project (2016-2019)
  - Keith Weber, Pablo Mejia-Ramos, Ruby Quea (Rutgers University)
  - Tim Fukawa-Connelly (Temple University)
  - Nick Wasserman, William McGuffey (Teachers College, Columbia University)
- Designed (12) "modules" to be incorporated into a Real Analysis course
  - Modules were *not* entirety of RA course
  - Modules were designed around specific RA definitions, theorems, proofs, etc.
  - Modules connecting RA content not just to secondary mathematics, but to situations in *teaching* secondary mathematics

Instructional Model

Traditional model

(if anything) "Make connections to school mathematics"

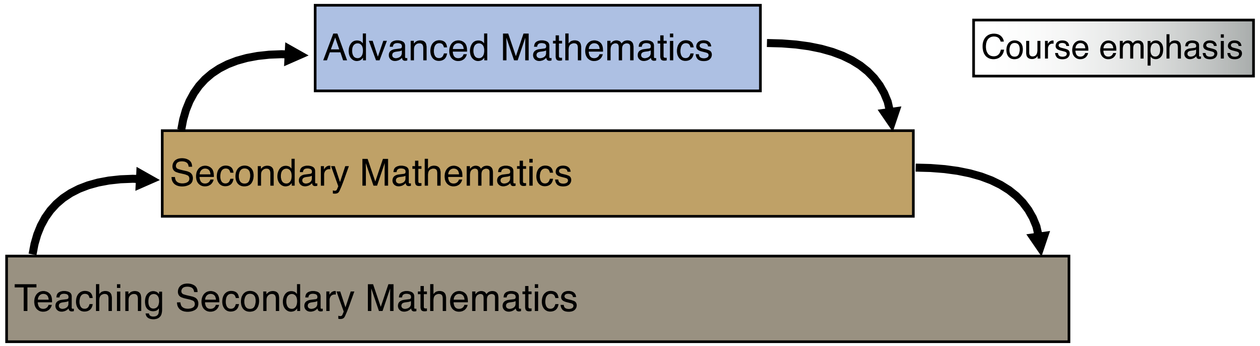


Trickle down effect: implicit hope is that a byproduct of learning advanced mathematics will be responding differently to instructional situations in the future.

Instructional Model

Alternate model

'Build up from' and 'Step down to' Teaching Practice



For Modules

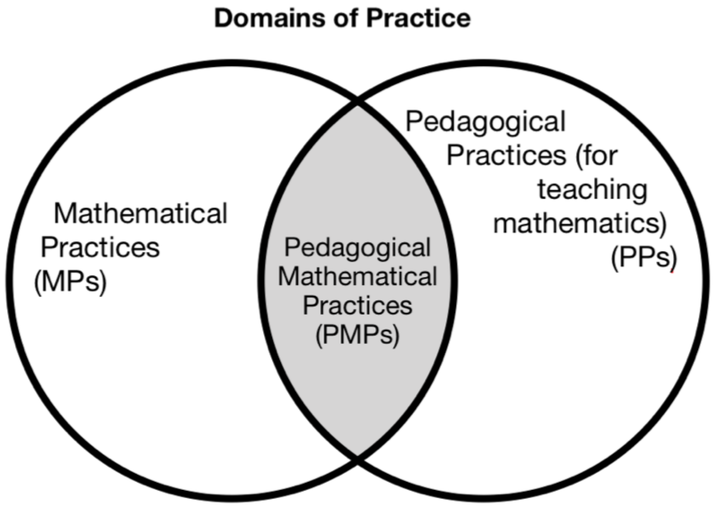
Begin with realistic situations in teaching secondary mathematics, where doing some facet of instruction well is well-situated to being learned in advanced mathematics.

Two comments

1. Advanced mathematics should be true to its rigorous form and reasoning (not a 'watered-down' version)
   - Mathematical goal and learning activity must include rigorous content
2. Responses to pedagogical situations should be improved in some meaningful way by learning the advanced mathematics.
   - Pedagogical goal and activity must be explicit; Pedagogical Mathematical Practices (PMPs)

Pedagogical Mathematical Practices

- Shulman (1986): PCK as intersection of domains of knowledge
- Pedagogical Mathematical Practices (PMPs): intersection of domains of practice (Wasserman, under review)
  - **Mathematical practices** are those activities in which mathematicians regularly engage (Cuoco et al., 2005; Rasmussen et al., 2005; SMPs)
  - **Pedagogical practices** (for teaching mathematics) are those activities in which mathematics teachers regularly engage (NCTM, 2014; HLPs)
  - **PMPs** are the kinds of practices that are common across both mathematicians and mathematics teachers – i.e., the kinds of actions, habits, lines of reasoning, etc., in which mathematicians and mathematics teachers regularly engage



Pedagogical Mathematical Practices

Pedagogical Aims (PMPs) from ULTRA (*not* an exhaustive list):

1. Acknowledge and revisit assumptions and mathematical constraints or limitations
2. Consider and use special cases to test and illustrate mathematical ideas
3. Expose logic as underpinning mathematical interpretation
4. Use simpler objects to study more complex objects
5. Avoid giving rules without accompanying mathematical explanation
6. Seek out multiple explanations

PMP Examples

Pedagogical Mathematical Practices

ULTRA 9

- **PMP.1**. Acknowledge and revisit assumptions and mathematical constraints or limitations
  - “*Attention to scope*” – the ‘domain’ for which a statement or argument holds
- **RA Content**: Proofs of derivative rules (Power rule, Product rule, Reciprocal/Quotient rule, Inverse function rule, Chain rule)

- How did we teach this content and discuss *pedagogical practice*?
  - We ordered the content in such a way so as to make explicit the *mathematical practice* of ‘attention to scope’
  - We then discussed secondary teaching situations for which this ‘attention to scope’ would be good *pedagogical practice*

Real Analysis

Derivative Proofs

- **Proof.** For which numbers sets for the power n (N,Z,Q,R) is this proof valid? If not valid for a set, at what step does the argument break down?

- Subsequently, introduced proofs of the product and quotient (reciprocal) rules, and that the power rule holds for **Z**
- Then, introduced proofs of the chain and inverse function rules, and that the power rule holds for **Q** and **R**

Prove: For $f(x)=x^n$, $f'(x)=nx^{n-1}$.

i. The difference of powers formula states that, given a real number, c:

$$\boxed{x^n - c^n = (x-c)\left(x^{n-1} + x^{n-2}\cdot c + \ldots + x\cdot c^{n-2} + c^{n-1}\right)}$$

ii. According to the definition, $f'(c)=\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$, when that limit exists. Substituting in the given function, for all $x \neq c$, we get:

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\frac{(x-c)\left(x^{n-1}+x^{n-2}\cdot c+\ldots+x\cdot c^{n-2}+c^{n-1}\right)}{x-c}=$$

$$=\lim_{x\to c}\left(x^{n-1}+x^{n-2}\cdot c+\ldots+x\cdot c^{n-2}+c^{n-1}\right)$$

iii. Therefore, since there are $\boxed{n \text{ terms}}$ on the right hand side, each of which approach c^{n-1} as x approaches c, $f'(c)=n\cdot c^{n-1}$ which is to say that $f'(x)=n\cdot x^{n-1}$.

Connecting to practice

Pedagogical Situation: ‘Explanations’

Mr. Ryan teaches everything from Pre-Algebra to Calculus. The following statements come from snapshots from his classes at different time during the year.

In an algebra class, Mr. Ryan is explaining exponents:

“Exponents are just repeated multiplication”

In a calculus class, Mr. Ryan is explaining the power rule:

“To take the derivative, you bring down the exponent to the front, and subtract one from the exponent.”

Attend to scope – acknowledge mathematical assumptions and limitations

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- **PMP.6.** Seek out multiple explanations
  - Explanations justify '*why',* not just 'that', something is true
- **RA Content:** Riemann integral/approximations; Proofs of integral properties (e.g., $\int(f+g)=\int f+\int g$, and $\int kf=k\int f$)
- How did we teach this content and discuss *pedagogical practice*?
  - We connected the content in such a way so as to make visual the implications on area – in order to engage *mathematical practice* of 'multiple explanations'
  - We then discussed secondary teaching situations for which having 'multiple explanation' approaches would be good *pedagogical practice*

Real Analysis

Integral Properties

- **Proof.** During proof (below), connected meaning to visualization, k∈$\mathbf{N}$, and to implication for **area**

To prove (ii) for the case $k \geq 0$, first verify that for any partition

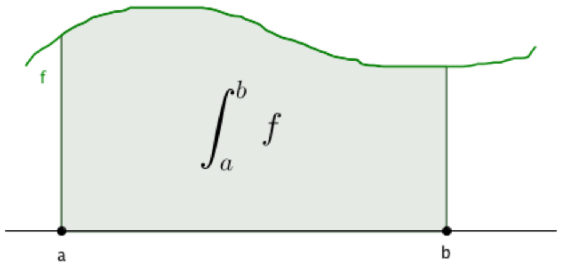
$$U(kf, P) = kU(f, P) \quad \text{and} \quad L(kf, P) = kL(f, P).$$

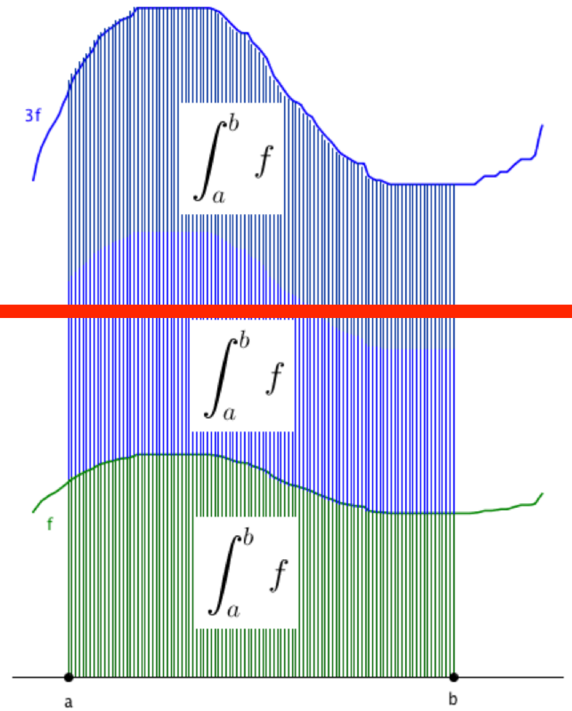
Exercise 1.3.5 is used here. Because f is integrable, there exist part[illegible] satisfying (1). Turning our attention to the function (kf), we see th[illegible]

$$\lim_{n\to\infty} [U(kf, P_n) - L(kf, P_n)] = \lim_{n\to\infty} k\,[U(f, P_n) - L(f, P_n)] = 0,$$

and the formula in (ii) follows. The case where $k < 0$ is similar except that we have

$$U(kf, P_n) = kL(f, P_n) \quad \text{and} \quad L(kf, P_n) = kU(f, P_n).$$





Cavalieri Principle (in 2D):
Vertical segments, between lines *x=a* and *x=b*, are congruent (by definition of 3f), so shapes have same area.

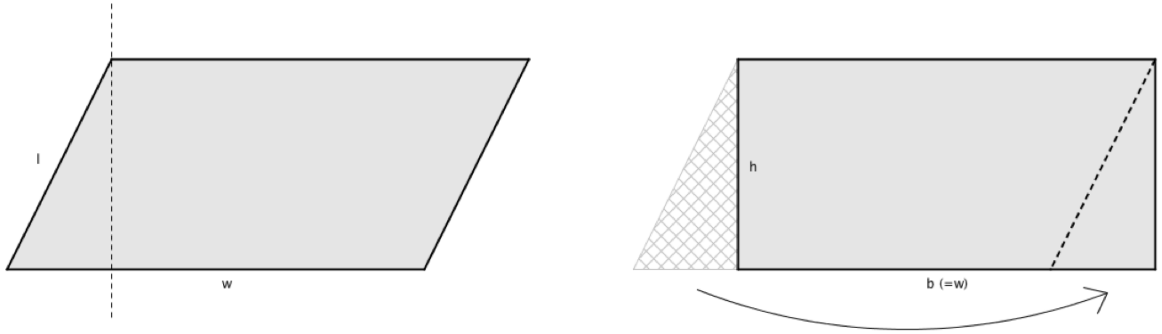
Connecting to practice

Pedagogical Situation: ‘Area-Justification’

“Cut-reassemble transformation”

(preserves area)

$$A_{par} = b*h$$



Connecting to practice

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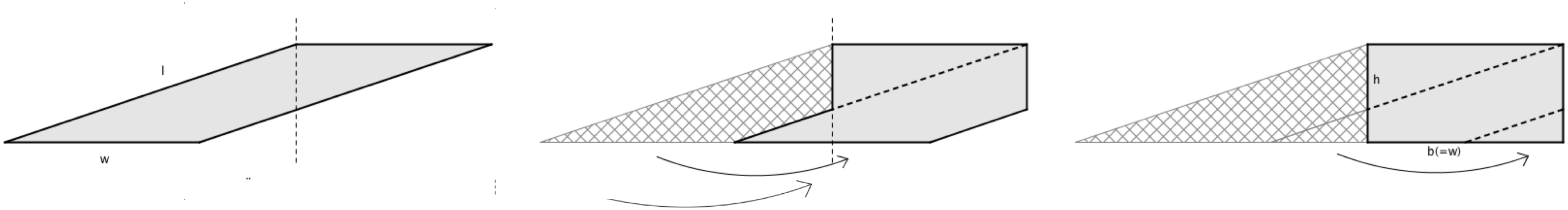
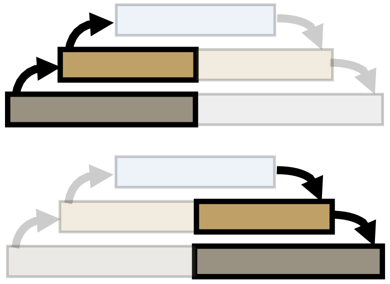


What to do when argument gets more complicated?



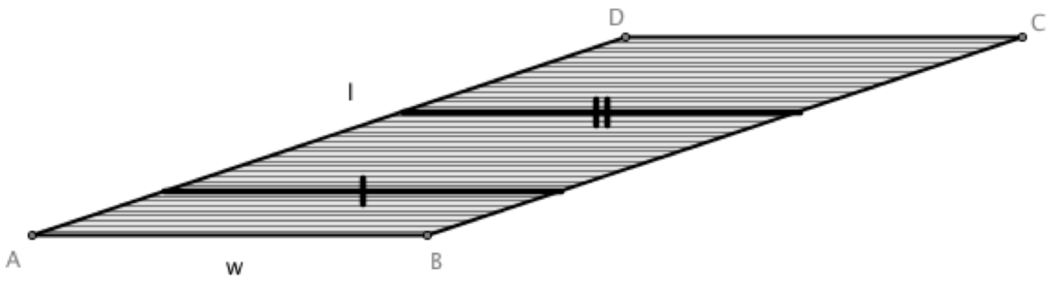
Connecting to practice

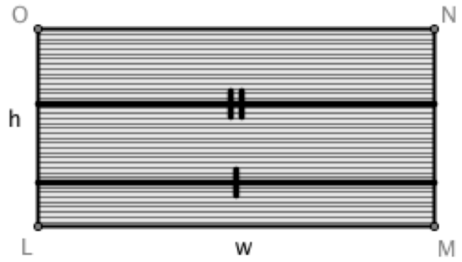
Pedagogical Situation: ‘Area-Justification’

“Cut-reassemble transformation”
(preserves area)

$$A_{par} = b*h$$

“Segment-skewing transformation”
(*also* preserves area)





Seek out *multiple explanations* – affords new ways of thinking about area, and justification

What to do when argument gets more complicated?

Conclusions

Summary of some ULTRA findings

Wasserman, N., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: Fostering mathematics teachers' 'attention to scope'. *Journal of Mathematics Teacher Education, 22*(4), 379-406.

- When *evaluating pedagogical quality*, PISTs increased the attention they gave to mathematical scope and limitations of a teacher's statements (PMP.1)
- PISTs valued the idea of attending to scope and language *for their teaching* (PMP.1)
- PISTs used the idea of attending to scope, being explicit with students about conditions and limitations, *in their own teaching* (PMP.1)

Summary of some ULTRA findings

Wasserman, N., & McGuffey, W. (accepted). Opportunities to learn from (advanced) mathematical coursework: A teacher-perspective in relation to observed classroom practice. *Journal for Research in Mathematics Education.*

- PISTs incorporated PMPs *into their own teaching practice*
- PISTs attributed some of their doing so to *learning activities in the ULTRA course*
- *Attributions* made by PISTs for various outcomes in their teaching *crossed*:
  - **PMPs** (ped goal) evident in their teaching were attributed to RA mathematical learning (math activity) (as well as ped activity of teaching situations)
  - **Mathematical outcomes** (math goal) evident in their teaching were attributed to discussing teaching situations (ped activity) (as well as math activity of learning RA)

Overall Conclusions

- Discussions about pedagogy do not often happen in university mathematics courses – because pedagogy is understood as a very separate issue
- In contrast, we have thought about *pedagogical* connections somewhat differently:
  - Mathematical practices evident in advanced mathematics can mirror pedagogical practice (*PMPs*)
- Being explicit about when mathematical practice mirrors pedagogical practice (PMPs) can help teachers connect mathematics learning to teaching secondary mathematics

Thanks!

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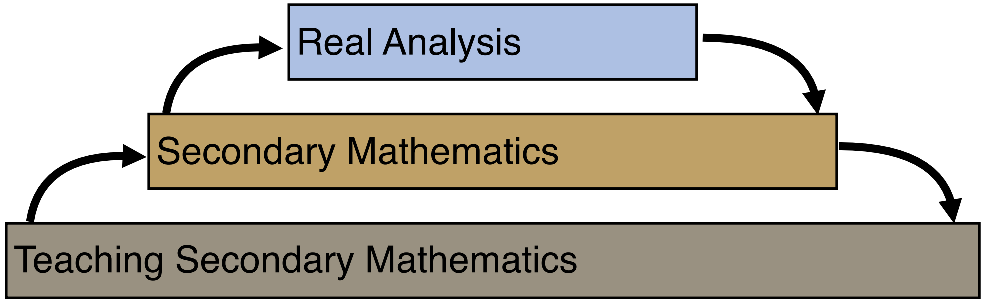
More about Project ULTRA



http://ultra.gse.rutgers.edu