Graded rings of rational twist in prime characteristic

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Let $d_1, \cdots, d_m$ pairwise relatively prime and $D = d_1 \cdots d_m$. There is a polynomial of degree $m-1$, $h_0(x)$, such that

$$h_0(n \cdot D) = |\{(i_1, i_2, \ldots, i_m) \in \mathbb{Z}_{\geq 0} : \sum_k i_k d_k = n \cdot D\}|.$$

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Are the coefficients of $h_0(x-1)$ always nonnegative?

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We will denote the minimal number of homogeneous generators of G_e as a subring of A over $A_0 = R$ by k_e.

Definition

The sequence $\{k_e\}_e$ is called the *growth* sequence for A. The *complexity* sequence is given by $\{c_{A,e} = k_{e+1} - k_e\}_e$.
The *complexity* of A is

$$\inf\{n > 0 : k_{e+1} - k_e = O(n^e)\}$$

and it is denoted by $\mathrm{cx}(A)$. It is obvious that $\mathrm{cx}(A) = 0$ when A is finitely generated as a ring over $A_0 = R$.

Definition

Let A be a $\mathbb{N}$-graded ring such that there exists a ring homomorphism $R \to A_0$, where R is a commutative ring. We say that A is a (left) *R-skew algebra* if $aR \subseteq Ra$ for all homogeneous elements $a \in A$.

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Corollary

Let A be a degree-wise finitely generated R-skew algebra such that $R = A_0$. Then $c_e(A)$ equals the minimal number of generators of $\frac{A_e}{(G_{e-1})_e}$ as a left R-module for all e.

Let E be the injective hull of k over R. Let $\mathscr{F}^e(E)$ be the collection of all additive maps $\phi : E \to E$ such that $\phi(r \cdot z) = r^{p^e}\phi(z)$, for all $r \in R, z \in E$.

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Definition

We define the *Frobenius complexity* of the ring R by

$$\mathrm{cx}_F(R) = \log_p(\mathrm{cx}(\mathscr{F}(E))).$$

Also, let $k_e^{\mathscr{F}} := k_e(\mathscr{F}(E))$, for all e and call these numbers the *Frobenius growth sequence* and $c_e = k_{e+1} - k_e$ defines the *complexity sequence*.

Let $\mathscr{R}$ be a $\mathbb{N}$-graded commutative ring of prime positive characteristic p with $\mathscr{R}_0 = R$. The following construction was introduced by Katzman, Schwede, Singh and W.Zhang.

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Define a ring structure on $T(\mathscr{R})$ by $a \in T_e, b \in T_{e'}$ such that

$$a * b = ab^{p^e}.$$

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$T_0 = R$ and if $a \in T_e, r \in R$, then $a * r = ar^{p^e} = r^{p^e}a = r^{p^e} * a$ and hence $T(\mathscr{R})$ is a skew R-algebra.

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Theorem (Katzman, Schwede, Singh, Zhang)

Let $(R, \mathfrak{m}, k)$ as above, E the R-injective hull of k. Then there exists a graded isomorphism:

$$\mathscr{F}(E) \simeq T(\mathscr{R}(\omega)).$$

Let $\mathscr{R}$ be an $\mathbb{N}$-graded commutative ring of prime characteristic p with $\mathscr{R}_0 = R$. As before, let $T(\mathscr{R}) := \oplus_{e \geq 0} \mathscr{R}_{p^e-1}$.
Denote the following:

- $T := T(\mathscr{R})$, $T_e := T_e(\mathscr{R}) = \mathscr{R}_{p^e-1}$.

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Let $\{c_e\}_{e\geq 0}$ equal the complexity sequence of T, and consider the generating function

$$\mathcal{C}_{\mathscr{R}}(z) = \mathcal{C}(z) := \sum_{e=0}^{\infty} c_e z^e = \sum_{e=1}^{\infty} c_e z^e \in \mathbb{Q}[[z]],$$

which we call the *twisted generating function* of $\mathscr{R}$.

Assume that $\mathscr{R}$ is finitely generated over $\mathscr{R}_0 = R$ and that R is Noetherian.

Question

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Is $\mathcal{C}(z)$ a rational function in z?

Josep Alvarez Montaner has also considered this question, in the cases studied in the literature.

Definition

We say that the grading on $\mathscr{R}$ has rational twist if $\mathcal{C}(z)$ is a rational function in z.

Let $\alpha_1, \ldots, \alpha_d$ complex numbers, $d \geq 1$ and $\alpha_d \neq 0$. Let $f : \mathbb{N} \to \mathbb{C}$ a function. The following assertions are equivalent:

1. $$\sum_{n \geq 0} f(n) x^n = \frac{P(x)}{Q(x)},$$
   where $Q(x) = 1 + \alpha_1 x + \ldots + \alpha_d x^d$ and $\deg(P(x)) < d$.

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2. For all $n \geq 0$
   $$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0$$
3. For all $n \geq 0$,
   $$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n,$$
   where $1 + \alpha_1 x + \ldots + \alpha_d x^d = \prod_{i=1}^{k}(1 - \gamma_i x)^{d_i}$, the γ_i's are distinct and nonzero and $P_i(n)$ is a polynomial of degree less than d_i.

Definition

We say that the grading on $\mathscr{R}$ has rational twist of dominant eigenvalue if $\mathcal{C}(z)$ is a rational function of the form $\frac{P(z)}{Q(z)}$ if P, Q do not have common roots, $Q(0) \neq 0$ and Q has a unique simple root of minimal magnitude, denoted by $1/\gamma$.

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Definition

If $c_e = c\gamma^e + \text{lower order terms } o(\gamma^e)$, then we call the number c the *twisted complexity multiplicity* of $\mathscr{R}$, or simply *t-multiplicity*.

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Definition

If $c_e = c\gamma^e + \text{lower order terms } o(\gamma^e)$, then we call the number c the *twisted complexity multiplicity* of $\mathscr{R}$, or simply *t-multiplicity*. If $\mathscr{R}$ is a graded ring of rational twist of dominant eigenvalue γ, then for $e \gg 0$, $c_e = c\gamma^e + \text{lower order terms } o(\gamma^e)$.

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Does $\mathscr{R}$ have grading of rational twist?

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For a graded ring $\mathscr{R}$ of rational twist, what is the meaning of the t-multiplicity?

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1. Let $\mathscr{R} = R[x_1, \ldots, x_m]$ graded with $\deg(x_i) = d_i$, $i = 1, \ldots, m$. Then the grading on $\mathscr{R}$ has rational twist.

Theorem

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1. Let $\mathscr{R} = R[x_1, \ldots, x_m]$ graded with $\deg(x_i) = d_i$, $i = 1, \ldots, m$. Then the grading on $\mathscr{R}$ has rational twist.
2. Let $\mathscr{R} = V_r(R[x_1, \ldots, x_m])$, where $r \geq 1$, be the rth Veronese subring of the standard graded polynomial ring $R[x_1, \ldots, x_m]$. Then the grading on $\mathscr{R}$ has rational twist.

Let $d_1, \ldots, d_m$ be relatively prime. Let $E = ord(p)$ in $\mathbb{Z}_D^\times$ and $D = gcd(d_1, \ldots, d_m)$. Let $\mathscr{R} = R[x_1, \ldots, x_m]$ graded with $\deg(x_i) = d_i$, $i = 1, \ldots, m$.

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Assume $h_0(x-1)$ has nonnegative coefficients and $E = 1, 2$. Assume $m \geq 3$ or $m = 2, D > 1$.

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Assume $h_0(x-1)$ has nonnegative coefficients and $E = 1, 2$. Assume $m \geq 3$ or $m = 2, D > 1$.

Then $\mathscr{R}$ has a grading of rational twist of dominant type with dominant eigenvalue $p^{m-2} < \gamma_p < p^{m-1}$ where $\lim_{p\to\infty} \frac{\gamma_p}{p^{m-1}} = 1 - \frac{1}{(m-1)!D}$ and $\lim_{p\to\infty} c_p = \frac{1}{(m-1)!D-1}$.