Generalized Gorenstein modules

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Definition

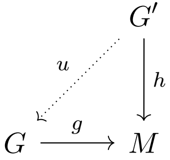
We say that a module $G \in Mod(R)$ is **Gorenstein projective** if there is an exact complex of projective modules $\mathbf{P} = \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \to \ldots$ such that $G = Z_0(P)$ and such that the complex stays exact when applying a functor $Hom(-, T)$, where T is any projective module (i.e. the complex $\ldots \to Hom(P_{-1}, T) \to Hom(P_0, T) \to Hom(P_1, T) \to \ldots$ is exact for any projective module T).

Any projective module P is Gorenstein projective $(0 \to P \xrightarrow{Id} P \to 0)$

Definition

We say that a module $M \in Mod(R)$ is **Gorenstein injective** if there is an exact complex of injective modules $\mathbf{I} = \ldots \to I_1 \to I_0 \to I_{-1} \to \ldots$ such that $M = Z_0(I)$ and such that the complex stays exact when applying a functor $Hom(A, -)$, where A is any injective module (i.e. the complex $\ldots \to Hom(A, I_1) \to Hom(A, I_0) \to Hom(A, I_{-1}) \to \ldots$ is exact for any injective module A).

A homomorphism $\phi : G \to M$ is a *Gorenstein projective precover* of M if G is Gorenstein projective and if for any Gorenstein projective module G' and any $\phi' \in Hom(G', M)$ there exists $u \in Hom(G', G)$ such that $\phi' = \phi u$.



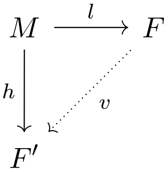
A precover $g : G \to M$ is said to be a *cover* if any homomorphism $u : G \to G$ such that $gu = g$, is an isomorphism.

A Gorenstein projective resolution of a module M is a complex

$$\ldots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M \to 0$$

such that $G_0 \to M$ and each $G_i \to Ker(G_{i-1} \to G_{i-2})$ for $i \geq 1$ are Gorenstein projective precovers.

A module M has a Gorenstein injective preenvelope if there exists a homomorphism $l : M \to F$ with F Gorenstein injective and such that for any Gorenstein injective module F', any homomorphism $h : M \to F'$ factors through l ($h = vl$ for some $v \in Hom(F, F')$).



A preenvelope l is said to be an envelope if it has one more property: any $v \in Hom(F, F)$ such that $vl = l$ is an automorphism of F.

Open question: the existence of the Gorenstein projective resolutions.
Generalizations of the Gorenstein modules - the Ding injective and Ding projective modules

- The *Ding projective modules* are the cycles of the exact complexes of projective modules that remain exact when applying a functor $Hom(-, F)$, with F any flat module.
- The *Ding injective modules* are the cycles of the exact complexes of injective modules that remain exact when applying a functor $Hom(A, -)$, with A any FP-injective module.

Open questions:

- is the class of Ding projectives, $\mathcal{DP}$, precovering over any ring?
- is the class of Ding injectives, $\mathcal{DI}$, preenveloping over any ring?

FP_n-injective and FP_n-flat modules

Definition

A module M is *n-finitely presented* (FP_n for short) if there exists an exact sequence $F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated free. A module M is FP_∞ if and only if $M \in FP_n$ for all $n \geq 0$.

$FP_0 \supseteqq FP_1 \supseteqq \ldots \supseteqq FP_n \supseteqq FP_{n+1} \supseteqq \ldots \supseteqq FP_\infty$, with FP_0 the class of all finitely generated modules, and FP_1 the finitely presented modules. A module M is FP_n-*injective* if $Ext^1_R(F, M) = 0$ for all $F \in FP_n$. From the definition, we get the following ascending chain:

$$Inj = \mathcal{FI}_0 \subseteq \mathcal{FI}_1 \subseteq \cdots \subset \mathcal{FI}_\infty.$$

A module N is FP_n-*flat* if $Tor_1(F, N) = 0$ for all $F \in FP_n$.

From the definition, we get the following ascending chain:

$$Flat = \mathcal{FF}_0 = \mathcal{FF}_1 \subseteq \mathcal{FF}_2 \subseteq \cdots \subset \mathcal{FF}_\infty.$$

Gorenstein FP_n-injective modules.

Definition. We say that a module $M \in Mod(R)$ is Gorenstein FP_n-injective if $M = Z_0(\mathbf{I})$ for some exact complex of injective modules $\mathbf{I} = \ldots \to I_1 \to I_0 \to I_{-1} \to \ldots$ that stays exact when applying a functor $Hom(A, -)$, where A is any FP_n-injective module. $\mathcal{GI}_n$ denotes the class of Gorenstein FP_n-injective modules.

By definition we get an ascending chain:
$\mathcal{GI}_\infty = \mathcal{GI}_{ac} \subseteq \cdots \subseteq \mathcal{GI}_2 \subseteq \mathcal{GI}_1 = \mathcal{DI} \subseteq \mathcal{GI}_0 = \mathcal{GI}$.

Dually, A module G is Gorenstein FP_n-projective if it a cycle in an exact complex of projective modules that remains exact when applying a functor $Hom(-, L)$ for any $L \in \mathcal{FF}_n$.
$\mathcal{GP}_n$ denotes the class of Gorenstein FP_n-projective modules.

Main results for Gorenstein FP_n-injectives

Theorem A Let R be any ring. For any $n \geq 1$, $({}^{\perp}\mathcal{GI}_n, \mathcal{GI}_n)$ is a hereditary cotorsion pair. In particular, $({}^{\perp}\mathcal{DI}, \mathcal{DI})$ is a hereditary cotorsion pair.

Theorem B Let R be any ring. For any $n \geq 2$, the class $\mathcal{GI}_n$ is enveloping.

Proposition If R is a coherent ring then $\mathcal{DI}$ is enveloping.

Main result for Gorenstein FP_n-projective modules:

Theorem C: Let R be any ring. For any $n \geq 2$, $\mathcal{GP}_n$ is a precovering class.

A sufficient condition for a class $\mathcal{C}$ be precovering is to be the left half of a complete cotorsion pair.

Recall $\mathcal{C}^{\perp} = \{M, Ext^1(C, M) = 0, \text{ for all } C \in \mathcal{C}\}$

and ${}^{\perp}\mathcal{C} = \{L, Ext^1(L, C) = 0, \text{ for all } C \in \mathcal{C}\}$

- A pair $(\mathcal{C}, \mathcal{L})$ is a *cotorsion pair* if $\mathcal{C}^{\perp} = \mathcal{L}$ and ${}^{\perp}\mathcal{L} = \mathcal{C}$.
- A cotorsion pair $(\mathcal{C}, \mathcal{L})$ is *complete* if for every M there are short exact sequences $0 \to L \to C \to M \to 0$ and $0 \to M \to L' \to C' \to 0$ with $C, C' \in \mathcal{C}$ and with $L, L' \in \mathcal{L}$.

A cotorsion pair $(\mathcal{C}, \mathcal{L})$ is *hereditary* if $Ext^i(C, L) = 0$ for any $C \in \mathcal{C}$, any $L \in \mathcal{L}$, all $i \geq 1$.

Examples: $(Proj, Mod)$, (Mod, Inj).

Theorem A For any $n \geq 1$, $({}^{\perp}\mathcal{GI}_n, \mathcal{GI}_n)$ is a hereditary cotorsion pair.
Known: $({}^{\perp}\mathcal{GI}_n, ({}^{\perp}\mathcal{GI}_n)^{\perp})$ is a cotorsion pair.

Proposition

Let $M \in ({}^{\perp}\mathcal{GI}_n)^{\perp}$. Then there is an exact complex $0 \to M \to E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} \ldots$ with each E^j injective and with $Ker f_j \in ({}^{\perp}\mathcal{GI}_n)^{\perp}$, for all j.

Proposition

Let $M \in ({}^{\perp}\mathcal{GI}_n)^{\perp}$. Then there is an exact complex $\ldots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} M \to 0$ with each E_j injective, and with $Ker(f_j) \in ({}^{\perp}\mathcal{GI}_n)^{\perp}$, for each $j \geq 0$.

Lemma

${}^{\perp}\mathcal{GI}_n = {}^{\perp\infty}\mathcal{GI}_n$, with ${}^{\perp\infty}\mathcal{GI}_n$ the class of modules A such that $Ext^i(A, G) = 0$ for all $G \in \mathcal{GI}_n$, and all $i \geq 1$.

Proof of Proposition 1.
$M \in \mathcal{GI} \bigcap \mathcal{FI}_n^{\perp}$.
- exact sequence $0 \to M \to E^0 \to M^0 \to 0$ with E^0 injective and $M^0 \in \mathcal{GI}$ (1).
- exact sequence $0 \to G \to D \to A \to 0$ with $D \in^{\perp} \mathcal{GI}$, $G \in \mathcal{GI}$ (2).
$A, D \in^{\perp} \mathcal{GI}$ implies $G \in^{\perp} \mathcal{GI}$.
(2) gives an exact sequence
$0 = Ext^1(G, M) \to Ext^2(A, M) \to Ext^2(D, M) = 0$ ($D \in^{\perp} \mathcal{GI}$, $M \in \mathcal{GI}$)
Thus $Ext^2(A, M) = 0$.
(1) gives an exact sequence
$0 = Ext^1(A, E^0) \to Ext^1(A, M^0) \to Ext^2(A, M) = 0$. So $M^0 \in (^{\perp}\mathcal{GI}_n)^{\perp}$.
Repeat with M replaced by M^0 to obtain an exact
$0 \to M^0 \to E^! \to M^1 \to 0$ with E^1 injective and $M^1 \in (^{\perp}\mathcal{GI}_n)^{\perp}$.
Continuing we obtain an exact complex $0 \to M \to E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} \dots$ with each E^j injective and with $Ker f_j \in (^{\perp}\mathcal{GI}_n)^{\perp}$, for all j.

Theorem A For any $n \geq 1$, $({}^{\perp}\mathcal{GI}_n, \mathcal{GI}_n)$ is a hereditary cotorsion pair.
Proof. It is known that $({}^{\perp}\mathcal{GI}_n, ({}^{\perp}\mathcal{GI}_n)^{\perp})$ is a cotorsion pair.
By Proposition 1 there is an exact complex
$0 \to M \to E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} \ldots$ with each E^j injective and with $Ker f_j \in ({}^{\perp}\mathcal{GI}_n)^{\perp}$, for all j. By Proposition 2, there s also an exact complex $\ldots \to E_1 \xrightarrow{g_1} E_0 \xrightarrow{g_0} M \to 0$ with each E_j injective, and with $Ker(g_j) \in ({}^{\perp}\mathcal{GI}_n)^{\perp}$, for each $j \geq 0$. Pasting them together we obtain an exact complex of injective modules,
$\ldots \to E_1 \to E_0 \to E^0 \to E^1 \to \ldots$ with all cycles in $({}^{\perp}\mathcal{GI}_n)^{\perp}$. This means that the complex stays exact when applying a functor $Hom(A, -)$ for any $A \in^{\perp} \mathcal{GI}_n$. In particular this is true for any $A \in \mathcal{FI}_n$. So all cycles are in $\mathcal{GI}_n$

To prove Theorem B:
To prove that $({}^{\perp}\mathcal{GI}_n, \mathcal{GI}_n)$ is complete for $n \geq 2$:

Theorem

Let $n \geq 2$, and let $\mathcal{F}$ denote the class of exact complexes of injective modules that stay exact when applying a functor $Hom(A, -)$ for any FP_n-injective module A. Then $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a complete cotorsion pair.

Proposition

Let X be a complex with $H_i(X) = 0$ for $i < 0$, and $X_i \in \mathcal{FI}_n$ for $i > 0$. Then $X \in^{\perp} \mathcal{F}$ if and only if $Z_0(X) \in^{\perp} \mathcal{GI}_n$.

Proposition

Let R be any ring. For any $n \geq 2$, $({}^{\perp}\mathcal{GI}_n, \mathcal{GI}_n)$ is a complete cotorsion pair.

Proof. There is an exact sequence $0 \to \overline{M} \to F \to X \to 0$. Applying $Z_0 = Hom_{Ch(R)}(\overline{R}, -)$ we obtain an exact sequence $0 \to M \to Z_0(F) \to Z_0(X) \to Ext^1(\overline{R}, \overline{M}) = 0$, with $Z_0(F) \in \mathcal{GI}_n$ and $Z_0(X) \in^{\perp} \mathcal{GI}_n$.

Theorem B Let R be any ring. For any $n \geq 2$, the class $\mathcal{GI}_n$ is enveloping.

Proposition

The following are equivalent:

1. The class of Gorenstein $\mathcal{FI}_n$-injective modules is enveloping.

2. The class ${}^{\perp}\mathcal{GI}_n$ is covering.

Proposition If R is a coherent ring then $\mathcal{DI}$ is an enveloping class.

Gorenstein FP_n-projective modules

Definition

Let $n \geq 1$ be an integer. A module G is Gorenstein FP_n-projective if it a cycle in an exact complex of projective modules that remains exact when applying a functor $Hom(-, L)$ for any $L \in \mathcal{FF}_n$.

We use $\mathcal{GP}_n$ to denote the class of Gorenstein $\mathcal{FP}_n$-projective modules.
- Since $\mathcal{FF}_1 = Flat$, so $\mathcal{GP}_0 = \mathcal{DP}$ (the Ding projective modules).
- And $\mathcal{FF}_\infty = Level$, so $\mathcal{GP}_\infty = \mathcal{GP}_{ac}$ (the Gorenstein AC-projective modules.

By definition we have an ascending chain

$$\mathcal{GP}_\infty = \mathcal{GP}_{ac} \subseteq \cdots \subseteq \mathcal{GP}_2 \subseteq \mathcal{GP}_1 = \mathcal{DP} \subseteq \mathcal{GP}.$$

Known: for $n \geq 2$, $M \in \mathcal{FF}_n \Leftrightarrow M^+ \in \mathcal{FI}_n$
and $C \in \mathcal{FI}_n \Leftrightarrow C^+ \in \mathcal{FF}_n$.

So, for $n \geq 2$, $(\mathcal{FI}_n, \mathcal{FF}_n)$ is a duality pair in the sense of Bravo - Gillespie - Hovey.

Theorem

(Bravo - Gillespie - Hovey) Let R be a ring and suppose $(\mathcal{C}, \mathcal{D})$ is a duality pair such that $\mathcal{D}$ is closed under pure quotients. Let P be a complex of projective modules. Then $A \otimes P$ is exact for all $A \in \mathcal{C}$ if and only if $Hom(P, N)$ is exact for all $N \in \mathcal{D}$.

Proposition

A module M is Gorenstein FP_n-projective if and only if there is an exact complex of projective modules $P = \ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{-1} \to \ldots$ such that $M = Z_0(P)$ and such that $A \otimes P$ is exact for all $A \in \mathcal{FI}_n$.

Definition

Let $\mathcal{B}$ be a class of right R-modules. We say that a module M is projectively coresolved Gorenstein $\mathcal{B}$-flat if $M = Z_0(P)$ for some $B \otimes -$-acyclic and exact complex P of projective modules.

- $\mathcal{PGF}_{\mathcal{B}}$ denotes the class of projectively coresolved Gorenstein $\mathcal{B}$-flat modules.

Theorem

(joint with Estrada and Perez) If $\mathcal{B}$ is a semi-definable class of right R-modules then $(\mathcal{PGF}_{\mathcal{B}}, \mathcal{PGF}_{\mathcal{B}}^{\perp})$ is a complete hereditary cotorsion pair. In particular, the class $\mathcal{PGF}_{\mathcal{B}}$ is precovering.

Since for any $n > 1$ the class of $\mathcal{FP}_n$-injective modules, $\mathcal{FI}_n$, is definable (so semi-definable also), and since $\mathcal{GP}_n = \mathcal{PGF}_{\mathcal{FI}_n}$, we obtain:

Theorem

(Theorem C) Let $n > 1$. The class of generalized Gorenstein projectives, $\mathcal{GP}_n$, is precovering.

References:

1. D. Bravo and J.Gillespie and M. Hovey: *The stable module category of a general ring*, **International Electronic Journal of Algebra**, 18:1–20, 2015.
2. D. Bravo, S. Estrada, A. Iacob. *FP_n-injective and FP_n-flat covers and preenvelopes and Gorenstein AC-flat covers*, **Alg. Colloquium**, vol 25, issue 2, pages 319 - 334, 2018.
3. S. Estrada, A. Iacob, M. Perez. *Model structures and relative Gorenstein flat modules and chain complexes*, chapter in **Contemporary Mathematics**, Volume 751, Print ISBN 978-1-4704-4368-9, Electronic ISBN: 978-1-4704-5608-5, pages 135 176.
4. A. Iacob: *Generalized Gorenstein modules*, submitted.